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Manifest Supersymmetry for BPS Walls in $\mathcal{N}=2$ Nonlinear Sigma Models

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Abstract

BPS equations and wall solutions are studied keeping (part of) supersymmetry (SUSY) manifest. Using $\mathcal{N}=1$ superfields, massive hyper-Kähler quotient is introduced to obtain massive $\mathcal{N}=2$ (8 SUSY) nonlinear sigma models in four dimensions with $T^*\mathbf{C}P^n$ target manifold, which yield BPS wall solutions for the $n=1$ case. We also describe massive hyper-Kähler quotient of $T^*\mathbf{C}P^n$ by using the harmonic superspace formalism which preserves all SUSY manifestly, and BPS equations and wall solutions are obtained in the $n=1$ case.

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1 Introduction

Brane world scenario has attracted much attention in recent years [1, 2]. To realize our world on a brane such as a wall, it is useful to consider supersymmetric (SUSY) theories which is the most promising theory beyond the standard model [3]. Wall (junction) configurations can preserve half (quarter) of the SUSY charges [4]–[8] and are called $\frac{1}{2}$ ($\frac{1}{4}$) BPS states. To obtain four dimensional brane as a wall/or junction, we have to consider a SUSY fundamental theory in five or more spacetime dimensions, and such a theory must have at least eight SUSY charges [9]. These SUSY are so restrictive that possible potential terms are severely constrained. The nontrivial interactions require either nonlinearity of kinetic term (nonlinear sigma model) or gauge interactions [10]–[17]. If the theory is dimensionally reduced to four dimensions, it has at least $\mathcal{N} = 2$ SUSY. In the $\mathcal{N} = 2$ SUSY in four dimensions, one has to consider nonlinear sigma model with nontrivial Kähler metric in field space if one wants interacting theories with only hypermultiplets. The target spaces of such theories must be hyper-Kähler (HK) manifolds [19]. Potential terms can be induced only when masses of hypermultiplets are introduced. It was shown that its form can be described by the norm of Killing vector of target metric [11, 13]. Therefore, we have to consider massive nonlinear sigma models for $\mathcal{N} = 2$ SUSY theories if one wants to obtain an interesting solution like domain walls using only the hypermultiplets.

There have been a number of works to study the nonlinear sigma models with eight supercharges [10]–[23]. The massive nonlinear sigma model with nontrivial Kähler metric as target space was studied, and BPS equations and BPS wall/or junction solutions were obtained [12]–[18]. Multi domain walls solution was also obtained and the dynamics of those walls was examined [15, 16]. In most papers, nonlinear sigma models were studied in terms of component fields. However, it is often useful to maintain as much SUSY as possible. For instance, maintaining three-dimensional SUSY is useful to obtain a low energy effective action (LEEA) on the wall in four-dimensional SUSY as fundamental theory [24]. Harmonic superspace formalism (HSF) [25] is most suited to maintain the SUSY maximally, but there has been relatively few attempt to formulate the BPS equations and to obtain BPS solutions in the HSF [27].

The purpose of our paper is to present SUSY formulation of BPS equations and solutions in $\mathcal{N} = 2$ nonlinear sigma models in four dimensions using both $\mathcal{N} = 1$ superfield and $\mathcal{N} = 2$ superfield (HSF). Action of nonlinear sigma model can be constructed by the method of HK quotient in both languages. Furthermore, one has to introduce masses for hypermultiplets in order to obtain a scalar potential with nontrivial interactions. Therefore, we need to construct a HK quotient for the massive nonlinear sigma model. We call such a quotient massive HK quotient. In this paper, we construct massive HK quotient of nonlinear sigma model on the cotangent bundle over $\mathbf{C}P^n$, namely $T^*\mathbf{C}P^n$, with the potential term and describe the action in both $\mathcal{N}=1$ and $\mathcal{N}=2$ formalisms. In $\mathcal{N}=1$ formalism, the massless HK sigma model on $T^*\mathbf{C}P^n$ was obtained as the HK quotient [22, 23]. The massive HK quotient was obtained in component level [16]. In $\mathcal{N}=2$ formalism, the massless model on $T^*\mathbf{C}P^1$ was first constructed in Ref. [28], and its central extension was analysed in Ref. [29]. The massive HK sigma model on $T^*\mathbf{C}P^n$ in superfield languages can be easily obtained as an extension of above models. Since $T^*\mathbf{C}P^n$ is one of the simplest classes of HK manifolds, we anticipate these quotient constructions to be useful in future.

In this paper, as the simplest case, we focus on the $n=1$ case, massive $T^*\mathbf{C}P^1$ model and examine the BPS equations and wall solutions. We show that the solution which is obtained from both languages corresponds to that derived in the component formalism [13]. Our formalism hopefully provides a starting point to obtain solutions and effective actions in more realistic cases of four-dimensional wall in five or higher dimensional theories.

Four is the minimum number of SUSY charges in four dimensions. Hence one might expect that one could deal with the models in the manner where four SUSY conserved by BPS wall is manifest, since BPS walls in $\mathcal{N}=2$ nonlinear sigma models conserve four SUSY. However, we cannot construct Lorentz invariant field theory keeping only those four SUSY conserved by the wall manifestly, because half SUSY condition for BPS wall itself breaks the Lorentz invariance for four dimensional spacetime. For instance, the Weyl spinor parameters of $\mathcal{N}=2$ SUSY ϵ^i $(i=1,2)$ are constrained by the $\frac{1}{2}$ SUSY condition for BPS wall solutions depending on a single coordinate y (with a simple choice of spinor basis and of parameters)

$$\sigma^2\bar{\epsilon}^1 = i\epsilon^1, \quad \sigma^2\bar{\epsilon}^2 = -i\epsilon^2 \tag{1.1}$$

as derived [1] in Appendix A.1. The four SUSY selected by this condition allows the model to be Lorentz invariant for only three dimensional spacetime (t,x,z) corresponding to the world volume of the wall.

In sect. 2, we introduce HK quotient methods in the $\mathcal{N}=1$ superfield formalism to construct massive nonlinear sigma models with $T^*\mathbf{C}P^n$ target space manifolds. We examine BPS equations and a wall solution in the $n=1$ case. In sect. 3, we study the same contents in the previous section by using the HSF. In sect. 4, the relation between the usual method to obtain LEEA of zero modes and the Manton's approach [30] is discussed. In Appendices we explain some details of Lorentz invariance versus $\frac{1}{2}$ BPS condition, HSF, and some topics related to each section.

[1] We follow mostly the notation of Ref. [9], except that $\mu,\nu,\ldots$ denote space time in four dimensions, $a,b\ldots$ three dimensions on the wall.

2 Massive hyper-Kähler quotient with $\mathcal{N}=1$ superfield

In this section, the massive HK sigma model on $T^*\mathbf{C}P^n$ is obtained in the $\mathcal{N}=1$ superfield formalism. We show that the potential term is the square of a tri-holomorphic Killing vector on the manifold as shown in the component level by Ref. [13]. We then obtain the BPS wall solution in the massive HK sigma model on $T^*\mathbf{C}P^1$. In appendix B, we give another $\mathcal{N}=1$ superfield formulation of the massive $T^*\mathbf{C}P^1$ model and its BPS wall solution, with generalisation to the Gibbons-Hawking metric [31] using the Hitchin method [32].

2.1 Massive HK sigma model on $T^*\mathbf{C}P^n$

In this subsection, we construct massive HK sigma model on $T^*\mathbf{C}P^n$ by using the HK quotient in $\mathcal{N}=1$ superfield formalism.

An $\mathcal{N}=2$ hypermultiplet can be decomposed into two chiral superfields in the $\mathcal{N}=1$ superfield formalism. We decompose $(n+1)$-hypermultiplets belonging to the fundamental representation of $SU(n+1)$ into $\mathcal{N}=1$ chiral superfields $\phi(x,\theta,\bar\theta)=(\phi^1,\cdots,\phi^{n+1})^T$ and $\chi(x,\theta,\bar\theta)=(\chi^1,\cdots,\chi^{n+1})^T$, belonging to the fundamental and anti-fundamental representations of $SU(n+1)$, respectively, whose transformation laws under $SU(n+1)$ are given by

$$\phi \to \phi' = g\phi\ , \quad \chi \to \chi' = (g^{-1})^T\chi\ , \tag{2.1}$$

with $g \in SU(n+1)$. An $\mathcal{N}=2$ vector superfield of the $U(1)$ gauge symmetry can be decomposed into $\mathcal{N}=1$ vector and chiral superfields, $V(x,\theta,\bar\theta)$ and $\sigma(x,\theta,\bar\theta)$. The $U(1)$-charges of ϕ and χ are 1 and -1, respectively. The $U(1)$ gauge transformation is given by

$$e^V \to e^{V'} = e^{i\Lambda - i\Lambda^\dagger}e^V\ , \quad \phi \to \phi' = e^{i\Lambda}\phi\ , \quad \chi \to \chi' = e^{-i\Lambda}\chi\ , \tag{2.2}$$

where $\Lambda(x,\theta,\theta)$ is a chiral superfield of a gauge parameter. Note that this $U(1)$ gauge symmetry is actually enhanced to its complexification, $U(1)^{\mathbf{C}}$. Then the Lagrangian of the $(n+1)$-hypermultiplets interacting with the auxiliary vector multiplet can be given by[2]

$$\begin{aligned}\mathcal{L} \ &= \ \int d^4\theta (e^V\phi^\dagger\phi + e^{-V}\chi^\dagger\chi - cV) + \left(\int d^2\theta\sigma(\chi^T\cdot\phi - b) + \text{c.c.}\right) \\ &\quad + \left(\int d^2\theta \sum_{\alpha=1}^{n} m_\alpha \chi^T H_\alpha \phi + \text{c.c.}\right)\ , \end{aligned}\tag{2.3}$$

[2]We omitted the trace part of the mass term $\chi^T\cdot\phi$, since it vanishes under the superspace integration using the constraint (2.6), in the following.

where the dot denotes the inner product of two vectors, H_α ($\alpha = 1, \cdots, n$) are the diagonal generators of the Cartan subalgebra of $SU(n+1)$ and m_α are the n mass parameters. Here $c \in \mathbf{R}$ and $b \in \mathbf{C}$ are coefficients of the Fayet-Iliopoulos (FI) term which transforms as an $SU(2)_R$ triplet (See Lagrangian (3.3) in the next section.) [34].

In the limit of $m_\alpha = 0$ for all α, the Lagrangian (2.3) becomes that of the massless HK nonlinear sigma model on $T^*\mathbf{C}P^n$ [20]–[23] whose isometry is $SU(n+1)$. By introducing the mass $m_\alpha \neq 0$ of the last term in (2.3), we will obtain the massive HK nonlinear sigma model on $T^*\mathbf{C}P^n$. The mass term explicitly breaks $SU(n+1)$ into $U(1)^n$: if we write $g = e^{i\varepsilon^A T_A}$ in (2.1) where T_A are the fundamental representation of the generators of $SU(n+1)$ ($A = 1, \cdots (n+1)^2 - 1$), the infinitesimal variation of the mass term

$$\delta_\varepsilon(\sum_{\alpha=1}^{n} m_\alpha \chi^T H_\alpha \phi) = i\chi^T [\sum_\alpha m_\alpha H_\alpha, \sum_A \varepsilon^A T_A]\phi \tag{2.4}$$

vanishes only when $T_A \sim H_\alpha$.

We eliminate the auxiliary fields V and σ to obtain the nonlinear Lagrangian. The equations of motion for V and σ read

$$\partial\mathcal{L}/\partial V = e^V|\phi|^2 - e^{-V}|\chi|^2 - c = 0\ , \tag{2.5}$$

$$\partial\mathcal{L}/\partial\sigma = \chi^T \cdot \phi - b = 0\ , \tag{2.6}$$

respectively. Setting $X = e^V$ in the first equation, we obtain the algebraic equation $|\phi|^2 X^2 - cX - |\chi|^2 = 0$, which can be solved to give $X = (c \pm \sqrt{c^2 + 4|\phi|^2|\chi|^2})/(2|\phi|^2)$. We thus obtain the Kähler potential of the form

$$K = \sqrt{c^2 + 4|\phi|^2|\chi|^2} - c\log\left(c + \sqrt{c^2 + 4|\phi|^2|\chi|^2}\right) + c\log|\phi|^2\ , \tag{2.7}$$

where we have chosen the plus sign of the solution for the positivity of the metric. This Kähler potential (2.7) is still invariant under the $U(1)^{\mathbf{C}}$ gauge transformation (2.2) up to a Kähler transformation. Fixing a gauge and substituting a solution of (2.6), we obtain the Lagrangian of the massive HK sigma model on $T^*\mathbf{C}P^n$ in terms of independent $\mathcal{N} = 1$ superfields.

We can consider the two cases of i) $c = 0$ and ii) $b = 0$, which are related by an $SU(2)_R$ transformation.

i) $c = 0$. Using the $U(1)^{\mathbf{C}}$ gauge degree of freedom, we can set [21, 22]

$$\phi = \frac{1}{\sqrt{1 + v^T \cdot w}} \begin{pmatrix} 1 \\ v \end{pmatrix}\ , \quad \chi = \frac{b}{\sqrt{1 + v^T \cdot w}} \begin{pmatrix} 1 \\ w \end{pmatrix}\ , \tag{2.8}$$

where v and w are vectors of n chiral superfields. We thus obtain the Lagrangian of the massive HK sigma model on $T^*\mathbf{C}P^n$ whose Kähler potential and superpotential are given by

$$
\begin{aligned}
K &= 2|b|\sqrt{\frac{(1+|v|^2)(1+|w|^2)}{|1+v^T\cdot w|^2}} , \\
W &= \frac{b}{1+v^T\cdot w}\sum_{\alpha=1}^{n} m_\alpha(1,w^T)H_\alpha \begin{pmatrix} 1 \\ v \end{pmatrix} ,
\end{aligned}
\tag{2.9}
$$

respectively.

ii) $b=0$. In this case, we can set

$$
\phi = \begin{pmatrix} 1 \\ v \end{pmatrix} , \quad \chi = \begin{pmatrix} -v^T\cdot w \\ w \end{pmatrix} ,
\tag{2.10}
$$

by using the $U(1)^{\mathbf{C}}$ gauge symmetry. We obtain

$$
\begin{aligned}
K &= \sqrt{c^2+4(1+|v|^2)(|v^T\cdot w|^2+|w|^2)} \\
&\quad -c\log\left(c+\sqrt{c^2+4(1+|v|^2)(|v^T\cdot w|^2+|w|^2)}\right)+c\log(1+|v|^2) , \\
W &= \sum_{\alpha=1}^{n} m_\alpha(-v^T\cdot w, w^T)H_\alpha \begin{pmatrix} 1 \\ v \end{pmatrix} .
\end{aligned}
\tag{2.11}
$$

Since $SU(2)_R$ rotates three complex structures among themselves, these two cases ($c=0$ and $b=0$) cannot be related to each other by a holomorphic field redefinition.

2.2 Massive HK sigma model on Eguchi-Hanson space

The $n=1$ case of $T^*\mathbf{C}P^n$, $T^*\mathbf{C}P^1$ is the Eguchi-Hanson space of the gravitational instanton [33]. The parameters b and c correspond to the blow up and the deformation of the orbifold singularity in $\mathbf{C}^2/\mathbf{Z}_2$, respectively. In this subsection, we present the explicit form of the Lagrangian in the $c=0$ case.

Setting $n=1$, $m_1\equiv\mu$ and $H_1=\frac{1}{2}\sigma_3$ the superpotential (2.9) becomes

$$
W = b\mu\frac{1}{1+vw} ,
\tag{2.12}
$$

where we have omitted a constant shift, since it disappears under the superspace integral.

We denote the coordinates of the sigma model manifold by the superfields $\varphi^i=(v,w)$. The Kähler metric $g_{ij^*}=\partial_i\partial_{j^*}K$, where ∂_i denotes a differential with respect to φ^i, is found to be

$$
g_{ij^*} = \frac{K}{4}\begin{pmatrix} \left(1+\frac{K^2}{4|b|^2}\right)\frac{1}{(1+|v|^2)^2} & -\frac{1}{|1+vw|^2}\frac{(w-v^*)^2}{(1+|v|^2)(1+|w|^2)} \\ -\frac{1}{|1+vw|^2}\frac{(v-w^*)^2}{(1+|v|^2)(1+|w|^2)} & \left(1+\frac{K^2}{4|b|^2}\right)\frac{1}{(1+|w|^2)^2} \end{pmatrix} ,
\tag{2.13}
$$

or

$$ds^2 = \frac{K}{4}\left(1+\frac{K^2}{4|b|^2}\right)\left[\frac{dvdv^*}{(1+|v|^2)^2}+\frac{dwdw^*}{(1+|w|^2)^2}\right] - \frac{K}{4|1+vw|^2}\frac{(v-w^*)^2dvdw^*+(w-v^*)^2dwdv^*}{(1+|v|^2)(1+|w|^2)} . \tag{2.14}$$

Since the determinant is $\det g_{ij^*} = |b|^2/|1+vw|^4$, the inverse of the metric is obtained as

$$g^{ij^*} = \frac{K|1+vw|^4}{4|b|^2}\begin{pmatrix} \left(1+\frac{K^2}{4|b|^2}\right)\frac{1}{(1+|w|^2)^2} & \frac{1}{|1+vw|^2}\frac{(v-w^*)^2}{(1+|v|^2)(1+|w|^2)} \\ \frac{1}{|1+vw|^2}\frac{(w-v^*)^2}{(1+|v|^2)(1+|w|^2)} & \left(1+\frac{K^2}{4|b|^2}\right)\frac{1}{(1+|v|^2)^2} \end{pmatrix} . \tag{2.15}$$

Therefore the scalar potential can be calculated as

$$\begin{aligned} V &= g^{ij^*}\partial_i W \partial_{j^*} W^* \\ &= \frac{|\mu|^2}{4}K\left[\frac{|w|^2}{(1+|w|^2)^2}+\frac{|v|^2}{(1+|v|^2)^2}+\frac{|v|^2+|w|^2}{(1+|v|^2)(1+|w|^2)}\right] , \end{aligned} \tag{2.16}$$

where we have used the same letters with superfields for their lowest components. The vacua are given by $|v| = |w| = 0$ or $|v| = |w| = \infty$.

Next, we show that this scalar potential can be rewritten by the norm of the Killing vector whose action preserves the superpotential, corresponding to the $SU(2)$ generator $\frac{1}{2}\sigma_3$. We note that the $SU(2)$ action (2.1) on ϕ and χ breaks the gauge fixing condition of $\chi_1/\phi_1 = b$ in Eq.(2.8). Hence a compensating $U(1)$ gauge transformation is needed for the $SU(2)$ action on v and w to preserve the gauge fixing condition. In the case of $g = e^{i\varepsilon\frac{1}{2}\sigma_3}$, the variation $\delta_\varepsilon(\chi_1/\phi_1) = i\varepsilon(\chi_1/\phi_1)$ should be compensated by (2.2) with $\Lambda = -\varepsilon/2$. Therefore we find $\delta_3 v \equiv \delta_\varepsilon v + \delta_\Lambda v = -i\varepsilon v$ and $\delta_3 w \equiv \delta_\varepsilon w + \delta_\Lambda w = i\varepsilon w$. We thus obtain the Killing vector for $\frac{1}{2}\sigma_3$, given by

$$k_3^i = \frac{1}{\varepsilon}\begin{pmatrix}\delta_3 v \\ \delta_3 w\end{pmatrix} = \begin{pmatrix} -iv \\ iw \end{pmatrix} . \tag{2.17}$$

Using this Killing vector, we find

$$V = |\mu|^2 g_{ij^*} k_3^i k_3^{*j} . \tag{2.18}$$

We can consider the projection map from the bundle $T^*\mathbf{C}P^1$ to the base manifold $\mathbf{C}P^1$. It is given by $v = w^*$ [21]. By this map, the metric (2.14) is mapped into the Fubini-Study metric on $\mathbf{C}P^1$

$$ds^2|_{v=w^*} = \frac{2|b|dvdv^*}{(1+|v|^2)^2} , \tag{2.19}$$

and the potential (2.16) is reduced to

$$V|_{v=w^*} = \frac{2|b||\mu|^2|v|^2}{(1+|v|^2)^2} , \tag{2.20}$$

which coincides with the one of the massive $\mathbf{C}P^1$ model [12].

2.3 BPS equation and its solution

In this subsection, we construct the BPS domain wall in the massive $T^*\mathbf{C}P^1$ model. The SUSY transformation on the fermion is given by

$$\delta_\epsilon \psi^i = i\sqrt{2}\sigma^\mu \bar{\epsilon}\partial_\mu \varphi^i + \sqrt{2}\epsilon F^i \ . \tag{2.21}$$

Let us choose $y = x^2$ as the spatial direction perpendicular to the BPS domain wall. Without loss of generality, we can require the direction of preserved SUSY as

$$e^{i\alpha}\sigma^2\bar{\epsilon} = i\epsilon \tag{2.22}$$

with a phase factor $e^{i\alpha}$ to be determined later. Then the BPS equations are given by [7]

$$\partial_2 \varphi^i = -e^{i\alpha} g^{ij^*}\partial_{j^*} W^* \ , \tag{2.23}$$

where the both sides are evaluated at classical fields. In the case of the massive $T^*\mathbf{C}P^1$ model, after eliminating the auxiliary fields, these BPS equations reduce to

$$\begin{aligned}
\partial_2 v &= e^{i\alpha}\frac{\mu^*}{4b}K(1+vw)^2\left[\frac{|1+vw|^2+(1+|v|^2)(1+|w|^2)}{|1+vw|^2(1+|w|^2)^2}w^* + \frac{(v-w^*)^2 v^*}{|1+vw|^2(1+|v|^2)(1+|w|^2)}\right], \\
\partial_2 w &= e^{i\alpha}\frac{\mu^*}{4b}K(1+vw)^2\left[\frac{|1+vw|^2+(1+|v|^2)(1+|w|^2)}{|1+vw|^2(1+|v|^2)^2}v^* + \frac{(w-v^*)^2 w^*}{|1+vw|^2(1+|v|^2)(1+|w|^2)}\right].
\end{aligned} \tag{2.24}$$

Now we must choose the phase $e^{i\alpha}$ to absorb the phase of the parameter [3] μ^*/b

$$e^{i\alpha}\frac{\mu^*}{b} = \left|\frac{\mu}{b}\right|. \tag{2.25}$$

By subtracting the complex conjugate of the second equation from the first one in Eq.(2.24), we obtain

$$\begin{aligned}
\frac{\partial(v-w^*)}{\partial y} &= \left|\frac{\mu}{b}\right|\frac{K}{4}\left[\left\{\left(\frac{1+vw}{|1+vw|}\right)^2 v^* - \left(\frac{1+v^*w^*}{|1+vw|}\right)^2 w\right\}\frac{(v-w^*)^2}{(1+|v|^2)(1+|w|^2)}\right. \\
&+ \left.\left\{\left(\frac{1+vw}{|1+vw|}\right)^2 \frac{w^*}{(1+|w|^2)^2} - \left(\frac{1+v^*w^*}{|1+vw|}\right)^2\frac{v}{(1+|v|^2)^2}\right\}\left\{|1+vw|^2+(1+|v|^2)(1+|w|^2)\right\}\right],
\end{aligned} \tag{2.26}$$

whose right-hand side vanishes for $v = w^*$. The BPS equation (2.26) dictates that $v = w^*$ is valid for arbitrary y, if an initial condition $v = w^*$ is chosen at some y. Since we can choose the initial condition $v = w^*$ at $y = -\infty$, we find the BPS equations (2.24) simply reduce to

$$\partial_2 v = |\mu| v \ , \tag{2.27}$$

[3]For simplicity, we choose μ to be real positive in the following.

which is the BPS equation on the submanifold $\mathbf{C}P^1$ (2.19) defined by $v = w^*$. Therefore we obtain a BPS wall configuration connecting two vacua $v = w^* = 0$ at $y = -\infty$ to $v = w^* = \infty$ at $y = \infty$ along $v = w^*$ with a constant phase $e^{i\varphi_0}$

$$v = w^* = e^{|\mu|(y+y_0)} e^{i\varphi_0} , \tag{2.28}$$

where y_0 is also a constant representing the position of the wall. Thus we find two collective coordinates (zero modes) corresponding to the spontaneously broken translation (y_0) and $U(1)$ symmetry (φ_0).

We can show that BPS solution (2.28) coincides with that derived in component formalism [13] through the following field redefinition [4] $v \to X, \varphi$

$$v \equiv e^{u+i\varphi}, \qquad X = |b| \tanh u, \tag{2.29}$$

where u, φ and X are real scalar fields. After the field redefinition, the theory of massive $\mathbf{C}P^1$ model is described by X and φ, and the wall solution (2.28) is mapped to

$$X = |b| \tanh |\mu|(y + y_0), \qquad \varphi = \varphi_0. \tag{2.30}$$

This solution coincides with that derived in Ref. [13].

3 HSF and domain wall solution

In this section, we describe the massive HK sigma model on $T^*\mathbf{C}P^n$ in the HSF, and examine BPS equations of the $n = 1$ case, massive $T^*\mathbf{C}P^1$ model.

As we discussed in the previous section, domain wall solutions can be obtained in the $T^*\mathbf{C}P^1$ case as (2.28). As in the $\mathcal{N} = 1$ SUSY theory, we can obtain the BPS equations from the SUSY transformations for fermions imposing half SUSY condition, and by eliminating auxiliary fields. The main difference between $\mathcal{N} = 1$ formalism and $\mathcal{N} = 2$ formalism (HSF) is that there is an infinite set of auxiliary fields in the harmonic superfield, while there is single auxiliary field F^i for each chiral superfield in the $\mathcal{N} = 1$ superfield formalism. As a result, BPS conditions contain an infinite set of auxiliary fields in addition to physical fields. To obtain the BPS equations, the infinite set of the auxiliary fields should be eliminated by using the solution of the equations of

[4] Actually, using this field redefinition, one can show that massive $\mathbf{C}P^1$ model corresponds to the truncated model of massive $T^*\mathbf{C}P^1$ model in component formalism given in Ref. [13]. We discuss the truncated model further in section 4.

motion for auxiliary fields. We call these solutions “on-shell condition”. After substituting the on-shell condition and the half SUSY condition into the SUSY transformation of fermions, the BPS equations can be obtained.

In the following, we first describe the action of the massive $T^*\mathbf{C}P^n$ model in the HSF, and briefly describe the $n = 1$ case. Next we derive the equations of motion for auxiliary fields and show how to eliminate the infinite set of auxiliary fields. Then, it is shown that the BPS equations are obtained by using the on-shell condition. Finally we solve the BPS equations and show that the solution coincides with Eq. (2.30). Notation we use in this section is summarized in Appendix C.1.

3.1 Massive HK sigma model on $T^*\mathbf{C}P^n$

We can easily describe the massive HK sigma model on $T^*\mathbf{C}P^n$ in the HSF by considering the action in terms of the $\mathcal{N} = 1$ superfield formalism (2.3). We consider $(n+1)$-hypermultiplets ϕ_a^+ $(a = 1, \ldots, n+1)$ in the fundamental representation of $SU(n+1)$ which transform under $U(1)$ gauge transformations with unit charge

$$\phi_a^+ \rightarrow e^{-i\lambda(\zeta_A,u)}\phi_a^+, \tag{3.1}$$

where λ is the real analytic superfield representing the gauge transformation parameter, and also the vector multiplet transforming under the $U(1)$ gauge transformation as

$$\delta V^{++} \;=\; D^{++}\lambda(\zeta_A, u). \tag{3.2}$$

Then, the action of the massive $T^*\mathbf{C}P^n$ model is described as

$$S = -\int d\zeta^{(-4)}du \sum_{a=1}^{n+1} \left\{ \widetilde{\phi_a^+}(D^{++} + iV^{++})\phi_a^+ + \xi^{++}V^{++} \right\}, \tag{3.3}$$

where $\xi^{++} = \xi^{(ij)}u^+_{(i}u^+_{j)}$ is the coefficient of the FI term which is the $SU(2)_R$ triplet. Harmonic variables are denoted as $u_i^\pm$, $i = 1, 2$ being $SU(2)_R$ indices. See Appendix C.1 for details. The integral measure $d\zeta^{(-4)}$ is the analytic measure which is defined by $d\zeta^{(-4)} = d^4x_A d^2\theta^+ d^2\bar{\theta}^+$. The covariant derivative D^{++} is defined as

$$D^{++} = \partial^{++} - 2i\theta^+\sigma^\mu\bar{\theta}^+\partial_\mu^A - (\theta^{+2}\bar{Z} - \bar{\theta}^{+2}Z), \tag{3.4}$$

where ∂^{++} is harmonic differential defined by $\partial^{++} = u_i^+ \frac{\partial}{\partial u_i^-}$, and ∂_μ^A is the spacetime derivative in analytic basis. The central charge is denoted as Z whose eigenvalue is given as [5]

$$Z\phi_a^+ = \sum_{\alpha=1}^{n} \sum_{b=1}^{n+1} m_\alpha (H_\alpha)_{ab} \phi_b^+, \tag{3.5}$$

where m_α is complex mass parameters, and H_α are the diagonal generators of the Cartan subalgebra of $SU(n+1)$ as in (2.3). The central charge Z vanishes for fields neutral under $SU(n+1)$ such as V^{++}. In the limit of $m_\alpha = 0$, the action (3.3) becomes massless $T^*\mathbf{C}P^n$ model whose isometry is $SU(n+1)$. The mass term explicitly breaks $SU(n+1)$ into $U(1)^n$. These features are identical to the case of $\mathcal{N} = 1$ formalism.

3.2 Massive HK sigma model on Eguchi-Hanson space

In the following, we focus on the $n = 1$ case, massive HK sigma model on $T^*\mathbf{C}P^1$. Here we follow the original notation introduced in Ref. [28], which uses $O(2)$ gauge invariant form instead of the $U(1)$ in Eq.(3.3). It is described by [6]

$$S = -\int d\zeta^{(-4)} du \left(\widetilde{q_1^+} D^{++} q_1^+ + \widetilde{q_2^+} D^{++} q_2^+ + V^{++}(\widetilde{q_1^+} q_2^+ - \widetilde{q_2^+} q_1^+ + \xi^{++}) \right), \tag{3.6}$$

where the central charge Z satisfies the following eigenvalue equation which is obtained by using field redefinition, and taking $n = 1, m_1 = \mu \in \mathbf{R}$ and $H_1 = \frac{1}{2}\sigma_3$ in (3.5)

$$Zq_a^+ = \frac{\mu}{2} q_a^+, \tag{3.7}$$

where we take the complex mass parameter μ to be real for simplicity . The action (3.6) is invariant under $O(2)$ gauge transformation

$$\delta q_1^+ = -\lambda(\zeta_A, u) q_2^+, \tag{3.8}$$
$$\delta q_2^+ = \lambda(\zeta_A, u) q_1^+, \tag{3.9}$$

and (3.2).

[5]Since the central charge is defined by $Z = -i(\partial_5 + i\partial_6)$, the solution of Eq. (3.5) depends on extra spacetime x_5 and x_6. But the action does not depend on them. See Ref. [26] in detail.

[6]The action (3.3) with $n = 1$ and (3.6) is related by $\phi_1^+ = \frac{1}{\sqrt{2}}(q_1^+ - iq_2^+)$, $\phi_2^{+\prime} = \frac{1}{\sqrt{2}}(q_1^+ + iq_2^+)$ with the identification $\widetilde{\phi_2^+} \equiv \phi_2^{+\prime}$ $(\phi_2^+ = -\widetilde{\phi_2^{+\prime}})$.

To write down the component action (3.6), we derive the equations of motion [7]. Varying (3.6) with respect to the superfields q_a^+, (and their conjugate) and V^{++} yield the equations of motion,

$$
\begin{aligned}
D^{++}q_1^+ + V^{++}q_2^+ &= 0, && (3.10)\\
D^{++}q_2^+ - V^{++}q_1^+ &= 0, && (3.11)\\
\widetilde{q_1^+}q_2^+ - \widetilde{q_2^+}q_1^+ + \xi^{++} &= 0, && (3.12)
\end{aligned}
$$

where (3.10) and (3.11) include kinematical and dynamical parts, and (3.12) is a constraint. The auxiliary fields are eliminated by using the solutions of the kinematical part of Eqs. (3.10) and (3.11). To derive the kinematical part of equations of motion, we substitute the component expansion (C.12) and (C.13) for the analytic superfields q_a^+ and V^{++} into (3.10) and (3.11) [8]. Then, one obtains the equations of motion as (C.15)-(C.28) for the Grassmann coefficients in (C.12) and (C.13), and one can solve easily the kinematical part (C.15)-(C.18) and (C.22)-(C.25). The solutions are given by

$$
\begin{aligned}
F_a^+(x_A,u) &= f_a^i(x_A)u_i^+, && (3.13)\\
\psi_a(x_A,u) &= \psi_a(x_A), \quad \bar{\varphi}_a(x_A,u) = \bar{\varphi}_a(x_A), && (3.14)\\
A_{1\mu}^-(x_A,u) &= 2(\partial_\mu^A f_1^i + V_\mu f_2^i)(x_A)u_i^-, && (3.15)\\
A_{2\mu}^-(x_A,u) &= 2(\partial_\mu^A f_2^i - V_\mu f_1^i)(x_A)u_i^-, && (3.16)\\
M_1^-(x_A,u) &= -\left(\bar{M}_v f_2^i - \frac{\mu}{2}f_1^i\right)(x_A)u_i^-, && (3.17)\\
M_2^-(x_A,u) &= \left(\bar{M}_v f_1^i + \frac{\mu}{2}f_2^i\right)(x_A)u_i^-, && (3.18)\\
N_1^-(x_A,u) &= -\left(M_v f_2^i + \frac{\mu}{2}f_1^i\right)(x_A)u_i^-, && (3.19)\\
N_2^-(x_A,u) &= \left(M_v f_1^i - \frac{\mu}{2}f_2^i\right)(x_A)u_i^-. && (3.20)
\end{aligned}
$$

Note that the infinite set of auxiliary fields in the harmonic expansion are eliminated and the physical fields f_a^i, ψ_a, $\bar{\varphi}_a$ and the Lagrange multipliers M_v, V_μ are left. The latter are eliminated by using algebraic equations as will be mentioned later.

At this stage, we can write down the component action. In the following, we focus on the bosonic part of the action in order to obtain the equations of motion for Lagrange multipliers which are necessary to derive the BPS equations [9]. Substituting (C.12), (C.13) and (3.13)-(3.20)

[7] In this section, we express the action using harmonic superfields with constraints instead of independent ones, in contrast to the $\mathcal{N}=1$ case (2.9) and (2.11) which were obtained after eliminating the Lagrange multipliers. Action can be expressed by independent harmonic superfields as in Refs. [26, 35]. However, we will solve the constraint and gauge away redundant degrees of freedom, after writing down the on-shell action.

[8] Here we take the Wess-Zumino gauge where the gauge transformation is not complexified but real (see (C.13)).

[9] We write down the full on-shell action including fermions in Appendix C.2.

into the action (3.6), and integrating Grassmann variables and the harmonic variable, the bosonic part of the action becomes

$$
\begin{aligned}
S_{\text{boson}} &= -\int d^4x_A \Big\{ (\partial^\mu_A f^i_1 + V^\mu f^i_2)(\partial^A_\mu \bar{f}_{1i} + V_\mu \bar{f}_{2i}) \\
&+(\partial^\mu_A f^i_2 - V^\mu f^i_1)(\partial^A_\mu \bar{f}_{2i} - V_\mu \bar{f}_{1i}) \\
&-\frac{1}{2}\left(\bar{M}_v \bar{f}^i_1 - \frac{\mu}{2}\bar{f}^i_2\right)\left(M_v f_{1i} - \frac{\mu}{2} f_{2i}\right) - \frac{1}{2}\left(\bar{M}_v \bar{f}^i_2 + \frac{\mu}{2}\bar{f}^i_1\right)\left(M_v f_{2i} + \frac{\mu}{2} f_{1i}\right) \\
&-\frac{1}{2}\left(M_v \bar{f}^i_1 + \frac{\mu}{2}\bar{f}^i_2\right)\left(\bar{M}_v f_{1i} + \frac{\mu}{2} f_{2i}\right) - \frac{1}{2}\left(M_v \bar{f}^i_2 - \frac{\mu}{2}\bar{f}^i_1\right)\left(\bar{M}_v f_{2i} - \frac{\mu}{2} f_{1i}\right) \\
&+\frac{1}{3} D_{v(ij)}(-\bar{f}^{(i}_1 f^{j)}_2 + \bar{f}^{(i}_2 f^{j)}_1 + \xi^{(ij)}) \Big\}. \qquad (3.21)
\end{aligned}
$$

Equations of motion for the auxiliary fields M_v and V^μ are given by

$$
\begin{aligned}
M_v &= -\bar{M}_v = -\frac{\mu}{2}\frac{(f^i_1 \bar{f}_{2i} - f^i_2 \bar{f}_{1i})}{f^i_1 \bar{f}_{1i} + f^i_2 \bar{f}_{2i}}, \qquad (3.22) \\
2V^\mu &= \frac{-(\partial^\mu_A \bar{f}_{1i} f^i_2 - \bar{f}_{1i}\partial^\mu_A f^i_2 - \partial^\mu_A \bar{f}_{2i} f^i_1 + \bar{f}_{2i}\partial^\mu_A f^i_1)}{f^i_1 \bar{f}_{1i} + f^i_2 \bar{f}_{2i}}. \qquad (3.23)
\end{aligned}
$$

Substituting (3.22) and (3.23) into (3.21), we finally obtain the bosonic part of the action

$$
\begin{aligned}
S_{\text{boson}} &= \int d^4x_A \Big(-\partial^\mu_A f^i_1 \partial^A_\mu \bar{f}_{1i} - \partial^\mu_A f^i_2 \partial^A_\mu \bar{f}_{2i} \\
&+\frac{(\partial^\mu_A \bar{f}_{1i} f^i_2 - \bar{f}_{1i}\partial^\mu_A f^i_2 - \partial^\mu_A \bar{f}_{2i} f^i_1 + \bar{f}_{2i}\partial^\mu_A f^i_1)^2}{4(f^i_1 \bar{f}_{1i} + f^i_2 \bar{f}_{2i})} \\
&-\frac{1}{3} D_{(ij)}(-\bar{f}^{(i}_1 f^{j)}_2 + \bar{f}^{(i}_2 f^{j)}_1 + \xi^{(ij)}) - V(f_1, f_2) \Big), \qquad (3.24) \\
V(f_1, f_2) &= \frac{\mu^2}{4}\frac{1}{f^i_1 \bar{f}_{1i} + f^i_2 \bar{f}_{2i}} \left\{ -|f^i_1 \bar{f}_{2i} - f^i_2 \bar{f}_{1i}|^2 + (f^i_1 \bar{f}_{1i} + f^i_2 \bar{f}_{2i})^2 \right\}. \qquad (3.25)
\end{aligned}
$$

It was proved that the target metric for the four independent bosonic fields is just the Eguchi-Hanson metric [20, 28, 35]. To see that the dimension of the physical boson manifold equals four, one should take into account that the constraint in (3.24)

$$
-\bar{f}^{(i}_1 f^{j)}_2 + \bar{f}^{(i}_2 f^{j)}_1 + \xi^{(ij)} = 0 \qquad (3.26)
$$

eliminates three out of the original eight bosonic degrees of freedom, while one more degree of freedom is gauged away by the $O(2)$ gauge invariance. In the end of this section, we explicitly show that the kinetic term in the action (3.24) corresponds to the nonlinear sigma model with Eguchi-Hanson metric.

Let us also note that the theory has discrete SUSY vacua [10]. After describing the potential (3.25) in terms of the four independent variables, it is found that the potential (3.25) corresponds to one which was originally derived in Ref. [13] (see (3.58)), and that there exist two SUSY discrete vacua. These SUSY vacua are understood as the fixed points of the Killing vector, as will be seen at the end of this section.

3.3 BPS equation and its solutions

In this subsection, we derive the BPS equations, but we give here only an outline how to derive the BPS equations. Detailed derivation is given in Appendix C.3.

In order to obtain the BPS equations, we have to derive the SUSY transformations for fermions. They can be derived in a model independent way although an infinite set of the auxiliary fields are involved. In our case, the SUSY transformations can be derived for $\lambda^-(\zeta_A, u)$ in the vector multiplet $V^{++}(\zeta_A, u)$, and $\psi_a(\zeta_A, u)$, $\bar{\varphi}_a(\zeta_A, u)$, $\xi_a^{--}(\zeta_A, u)$ and $\bar{\chi}_a^{--}(\zeta_A, u)$ in the hypermultiplets $q_a^+(\zeta_A, u)$, since the action (3.6) is described by those superfields. However, we do not have to derive the SUSY transformations for all fermionic components because most fermions are auxiliary fields. Actually, since the bosonic part of the action (3.24) is described by the on-shell component $f_a^i(x_A)$, it is enough to derive the SUSY transformations for their superpartners $\psi_a(x_A)$ and $\bar{\varphi}_a(x_A)$ which are the first order components in Grassmann expansion of the analytic superfields (C.12) [11].

Recall that the Wess-Zumino gauge is chosen in our case. Since SUSY transformations break the Wess-Zumino gauge, one has to pull back to the Wess-Zumino gauge by using the $O(2)$ gauge transformation (3.2). As a result, SUSY transformation $\hat{\delta}$ in the Wess-Zumino gauge is defined as $\hat{\delta} = \delta_S + \delta_G$ where δ_S and δ_G are the original SUSY transformation and $O(2)$ gauge transformation, respectively, in order to preserve the Wess-Zumino gauge. The gauge parameters $\lambda(\zeta_A, u)$ in $O(2)$ gauge transformation are determined so as to keep the Wess-Zumino gauge ((C.42)-(C.47)). Putting altogether, the SUSY transformations for on-shell fermions are derived as (C.54)-(C.57). Substituting the on-shell condition (3.13)-(3.20) and the half SUSY

[10]The scalar potential (3.25) was originally derived in Ref. [29]. It was argued that supersymmetry is spontaneously broken in contrast to our result of partial SUSY conservation. As is shown at the end of this section, there exist two discrete SUSY vacua in the theory.

[11]Since the scalars $f_a^i(x_A)$ are $SU(2)_R$ doublets in the Fayet-Sohnius hypermultiplets, their superpartners are $SU(2)_R$ singlet, $\psi_a(x_A)$ and $\bar{\varphi}_a(x_A)$. Actually, it is found that full on-shell action is described by $f_a^i(x_A)$, $\psi_a(x_A)$ and $\bar{\varphi}_a(x_A)$ (see Appendix C.2). Alternatively, it is found that the fields ξ_a^{--} and $\bar{\chi}_a^{--}$ are auxiliary fields since they do not have the kinetic term (see the equations of motion (C.19)-(C.21) and (C.26)-(C.28)).

condition (1.1) into (C.54)-(C.57), we obtain the BPS equations,

$$\begin{aligned}
\hat{\delta}\psi_1 &= \sqrt{2}\epsilon_1\left\{(\bar{M}_v - V_2)f_2^1 - \left(\frac{\mu}{2}+\partial_2^A\right)f_1^1\right\} \\
&\qquad + \sqrt{2}\epsilon_2\left\{(\bar{M}_v + V_2)f_2^2 - \left(\frac{\mu}{2}-\partial_2^A\right)f_1^2\right\} = 0, \qquad (3.27)\\
\hat{\delta}\bar{\varphi}_1 &= \sqrt{2}\bar{\epsilon}_1\left\{(M_v + V_2)f_2^1 + \left(\frac{\mu}{2}+\partial_2^A\right)f_1^1\right\} \\
&\qquad + \sqrt{2}\bar{\epsilon}_2\left\{(M_v - V_2)f_2^2 + \left(\frac{\mu}{2}-\partial_2^A\right)f_1^2\right\} = 0, \qquad (3.28)\\
\hat{\delta}\psi_2 &= \sqrt{2}\epsilon_1\left\{-(\bar{M}_v - V_2)f_1^1 - \left(\frac{\mu}{2}+\partial_2^A\right)f_2^1\right\} \\
&\qquad + \sqrt{2}\epsilon_2\left\{-(\bar{M}_v + V_2)f_1^2 - \left(\frac{\mu}{2}-\partial_2^A\right)f_2^2\right\} = 0, \qquad (3.29)\\
\hat{\delta}\bar{\varphi}_2 &= \sqrt{2}\bar{\epsilon}_1\left\{-(M_v + V_2)f_1^1 + \left(\frac{\mu}{2}+\partial_2^A\right)f_2^1\right\} \\
&\qquad + \sqrt{2}\bar{\epsilon}_2\left\{-(M_v - V_2)f_1^2 + \left(\frac{\mu}{2}-\partial_2^A\right)f_2^2\right\} = 0. \qquad (3.30)
\end{aligned}$$

To satisfy the BPS equations (3.27)-(3.30), all coefficients of ϵ_i must vanish, namely there are eight BPS equations. However, using the relation $M_v = -\bar{M}_v$ (see (3.22)), we find that only four equations are independent;

$$(M_v + V_2)f_2^1 + \left(\frac{\mu}{2}+\partial_2^A\right)f_1^1 = 0, \qquad (3.31)$$

$$(M_v - V_2)f_2^2 + \left(\frac{\mu}{2}-\partial_2^A\right)f_1^2 = 0, \qquad (3.32)$$

$$-(M_v + V_2)f_1^1 + \left(\frac{\mu}{2}+\partial_2^A\right)f_2^1 = 0, \qquad (3.33)$$

$$-(M_v - V_2)f_1^2 + \left(\frac{\mu}{2}-\partial_2^A\right)f_2^2 = 0. \qquad (3.34)$$

To solve the BPS equations (3.31)-(3.34), we first have to solve the constraint (3.26) and gauge away the $O(2)$ gauge degrees of freedom. To do that we first set the following parameterisation

$$\phi_1^\alpha = \frac{1}{\sqrt{2}}(f_1^{2,\alpha} + i f_2^{2,\alpha}), \qquad \phi_2^\alpha = \frac{1}{\sqrt{2}}(f_1^{1,\alpha} + i f_2^{1,\alpha}), \qquad (3.35)$$

where $\alpha = 1, 2$, and $f_a^{i,1} = f_a^i$ and $f_a^{i,2} = \bar{f}_a^i$. In this basis, the action corresponds to that given by Curtright and Freedman [20]. The $SU(2)_R$ transformation allows us to choose $\xi^{(11)} = \xi^{(22)} = 0$ and $\xi^{(12)} = -i\xi$ ($\xi \in \mathbf{R}, \xi > 0$) without loss of generality [12]. It is most convenient to introduce independent fields $z^\alpha, \bar{z}^\alpha,\ \alpha = 1, 2$ through the following Ansatz [21, 35]

$$\phi_1^\alpha = g(r)\frac{z^\alpha}{\sqrt{r}}, \qquad \phi_2^\alpha = f(r) i\sigma^{2\alpha\beta}\frac{\bar{z}^\beta}{\sqrt{r}}, \qquad (3.36)$$

[12] Taking a $SU(2)_R$ transformation on the model of this case, we can obtain the case of $\xi^{(11)} \neq 0$ and $\xi^{(22)} \neq 0$.

with the constraint

$$\phi_1^1\phi_2^2 - \phi_1^2\phi_2^1 = -z^1\bar{z}^1 - z^2\bar{z}^2, \tag{3.37}$$

where z^α are complex fields and $r = z^1\bar{z}^1 + z^2\bar{z}^2$. The real functions $f(r)$ and $g(r)$ are uniquely determined by the constraints (3.26) and (3.37) as

$$f(r)^2 = -\xi + \sqrt{r^2+\xi^2}, \qquad g(r)^2 = \xi + \sqrt{r^2+\xi^2}. \tag{3.38}$$

At this stage, the action can be described by the independent complex fields z^α. Finally we set for later convenience [13],

$$z^1 = \sqrt{r}\cos\tfrac{\Theta}{2}\exp\tfrac{i}{2}(\Psi+\Phi), \tag{3.39}$$

$$z^2 = \sqrt{r}\sin\tfrac{\Theta}{2}\exp\tfrac{i}{2}(\Psi-\Phi), \tag{3.40}$$

$$0\le r\le\infty, \quad 0\le\Theta\le\pi, \quad 0\le\Phi\le 2\pi, \quad 0\le\Psi\le 2\pi.$$

By using the above spherical coordinates, the BPS equations are rewritten as

$$\begin{aligned} 0 &= -\frac{e^{\frac{i}{2}(\Phi-\Psi)}}{2\sqrt{2}\sqrt{r^2+\xi^2}}\Bigg\{r'(f-g)\sin\frac{\Theta}{2} + \xi\cos\frac{\Theta}{2}(f+g)(\mu\sin\Theta + i\sin\Theta\Phi' + \Theta') \\ &+ \quad r(f-g)\left(\mu\sin\frac{\Theta}{2} + i\sin\frac{\Theta}{2}\Phi' - i\sin\frac{\Theta}{2}\Psi' + \cos\frac{\Theta}{2}\Theta'\right)\Bigg\}, \end{aligned} \tag{3.41}$$

$$\begin{aligned} 0 &= \frac{e^{\frac{i}{2}(\Phi+\Psi)}}{2\sqrt{2}\sqrt{r^2+\xi^2}}\Bigg\{r'(f-g)\cos\frac{\Theta}{2} - \xi\sin\frac{\Theta}{2}(f+g)(\mu\sin\Theta - i\sin\Theta\Phi' + \Theta') \\ &- \quad r(f-g)\left(\mu\cos\frac{\Theta}{2} - i\cos\frac{\Theta}{2}\Phi' - i\cos\frac{\Theta}{2}\Psi' + \sin\frac{\Theta}{2}\Theta'\right)\Bigg\}, \end{aligned} \tag{3.42}$$

$$\begin{aligned} 0 &= -\frac{ie^{\frac{i}{2}(\Phi-\Psi)}}{2\sqrt{2}\sqrt{r^2+\xi^2}}\Bigg\{r'(f+g)\sin\frac{\Theta}{2} - \xi\cos\frac{\Theta}{2}(f-g)(\mu\sin\Theta + i\sin\Theta\Phi' + \Theta') \\ &+ \quad r(f+g)\left(\mu\sin\frac{\Theta}{2} + i\sin\frac{\Theta}{2}\Phi' - i\sin\frac{\Theta}{2}\Psi' + \cos\frac{\Theta}{2}\Theta'\right)\Bigg\}, \end{aligned} \tag{3.43}$$

$$\begin{aligned} 0 &= -\frac{ie^{\frac{i}{2}(\Phi+\Psi)}}{2\sqrt{2}\sqrt{r^2+\xi^2}}\Bigg\{r'(f+g)\cos\frac{\Theta}{2} + \xi\sin\frac{\Theta}{2}(f-g)(\mu\sin\Theta - i\sin\Theta\Phi' + \Theta') \\ &- \quad r(f+g)\left(\mu\cos\frac{\Theta}{2} - i\cos\frac{\Theta}{2}\Phi' - i\cos\frac{\Theta}{2}\Psi' + \sin\frac{\Theta}{2}\Theta'\right)\Bigg\}, \end{aligned} \tag{3.44}$$

where prime denotes the derivative in terms of y. BPS wall solution should approach the supersymmetric discrete vacua as $y\to\pm\infty$. By eliminating the terms with y derivative in Eqs. (3.41)-(3.44), we find the supersymmetric vacuum condition

$$0 = \xi\cos\frac{\Theta}{2}(f+g)\mu\sin\Theta + r(f-g)\mu\sin\frac{\Theta}{2}, \tag{3.45}$$

[13] About the domain of the coordinates, see Ref. [33].

$$
\begin{aligned}
0 &= \xi \sin\frac{\Theta}{2}(f+g)\mu\sin\Theta + r(f-g)\mu\cos\frac{\Theta}{2}, &(3.46)\\
0 &= \xi \cos\frac{\Theta}{2}(f-g)\mu\sin\Theta - r(f+g)\mu\sin\frac{\Theta}{2}, &(3.47)\\
0 &= \xi \sin\frac{\Theta}{2}(f-g)\mu\sin\Theta - r(f+g)\mu\cos\frac{\Theta}{2}. &(3.48)
\end{aligned}
$$

It is found that there are only two vacua satisfying all these equations : $(r, \Theta) = (0,0), (0,\pi)$. Therefore we consider the domain wall solution connects these vacua, and we can expect that Θ has nontrivial configuration. After some calculations, we can derive the four independent differential equations from Eqs. (3.41)-(3.44) in the following

$$
\begin{aligned}
r' &= \mu\cos\Theta\cdot r, \qquad r\cdot\Psi' = 0, &(3.49)\\
\Theta' &= -\mu\sin\Theta, \qquad \Phi' = 0. &(3.50)
\end{aligned}
$$

The boundary condition of $r = 0$ at $y = -\infty$ dictates the solution [14] of (3.49) to be $r = 0$ and $\Psi = 0$. On the other hand, nontrivial BPS solutions are obtaind from (3.50) as

$$
\Theta = \arccos[\tanh\mu(y+y_0)], \qquad \Phi = \varphi_0, \tag{3.51}
$$

where y_0 and φ_0 are real constants: y_0 determines the position of the domain wall along y direction and φ_0 corresponds to the Nambu-Goldstone (NG) mode of $U(1)$ isometry of target space.

We can show that the solution (3.51) can be mapped to (2.30). To see this, we put the following parameterization [15]

$$
\begin{aligned}
X^1 &= r\sin\Theta\cos\Psi, &(3.52)\\
X^2 &= r\sin\Theta\sin\Psi, &(3.53)\\
X^3 &= \sqrt{r^2+\xi^2}\cos\Theta, &(3.54)\\
\varphi &= \Phi+\Psi. &(3.55)
\end{aligned}
$$

Substituting $r = \Psi = 0$ and (3.51) into (3.52)-(3.55), we obtain the following form

$$
\begin{aligned}
X^1 &= X^2 = 0, &(3.56)\\
X^3 &= \xi\tanh\mu(y+y_0), \qquad \varphi = \varphi_0. &(3.57)
\end{aligned}
$$

It is found that this solution coincides with that derived in Ref. [13] [16]. We also show that the bosonic part of the action (3.24) corresponds to that given in Ref. [13]. By using the parame-

[14] We may set to $\Psi = 0$ because of the singularity of coordinate Ψ at $r = 0$ on the target space.

[15] This parameterization has an ambiguity in the rotation of X^1, X^2 and X^3.

[16] We find the *same* BPS wall solution as previous section in spite of solving the BPS equation associated with *different* $\frac{1}{2}$ SUSY condition. The relation between two $\frac{1}{2}$ SUSY conditions are discussed in Appendix A.

terisation (3.35)-(3.40) and (3.52)-(3.55), the bosonic part of the action (3.24) can be rewritten as

$$\mathcal{L} = -\frac{1}{2}\left\{U\partial_\mu \mathbf{X}\cdot\partial^\mu \mathbf{X} + U^{-1}\mathcal{D}_\mu\varphi\mathcal{D}^\mu\varphi + \mu^2 U^{-1}\right\}, \tag{3.58}$$

where $\mathcal{D}_\mu\varphi = \partial_\mu\varphi + \mathbf{A}\cdot\partial_\mu\mathbf{X}$ and

$$\boldsymbol{\nabla}\times\mathbf{A} = \boldsymbol{\nabla}U. \tag{3.59}$$

The harmonic function U can be described

$$U = \frac{1}{2}\left[\frac{1}{|\mathbf{X}-\xi\mathbf{n}|} + \frac{1}{|\mathbf{X}+\xi\mathbf{n}|}\right], \tag{3.60}$$

where $\mathbf{n}$ is a unit three vector, which is given by $\mathbf{n} = (0,0,1)$. $\mathbf{A}$ is a potential whose solution is given as

$$\begin{aligned}
A_1 &= \frac{1}{2}\left\{\frac{X^2}{|\mathbf{X}-\xi\mathbf{n}|(X^3-\xi+|\mathbf{X}-\xi\mathbf{n}|)} + \frac{X^2}{|\mathbf{X}+\xi\mathbf{n}|(X^3+\xi+|\mathbf{X}-\xi\mathbf{n}|)}\right\}, && (3.61)\\
A_2 &= \frac{1}{2}\left\{\frac{-X^1}{|\mathbf{X}-\xi\mathbf{n}|(X^3-\xi+|\mathbf{X}-\xi\mathbf{n}|)} + \frac{-X^1}{|\mathbf{X}+\xi\mathbf{n}|(X^3+\xi+|\mathbf{X}+\xi\mathbf{n}|)}\right\}, && (3.62)\\
A_3 &= 0. && (3.63)
\end{aligned}$$

It is found that the target metric of the action (3.58) is just the Eguchi-Hanson metric [20, 28, 35].

Finally we give the BPS solution in terms of harmonic superfields q_a^+. This is a starting point to derive the low energy effective action (LEEA) around the wall background. They are derived by using the change of variables (3.35)-(3.40). The results are

$$\begin{aligned}
q_1^+ = f_1^i u_i^+ &= \sqrt{\frac{\xi}{2}}e^{\frac{i}{2}\varphi_0}\begin{pmatrix} -\sqrt{1-\tanh(\mu(y+y_0))}u_1^+ \\ \sqrt{1+\tanh(\mu(y+y_0))}u_2^+ \end{pmatrix}, && (3.64)\\
q_2^+ = f_2^i u_i^+ &= -i\sqrt{\frac{\xi}{2}}e^{\frac{i}{2}\varphi_0}\begin{pmatrix} \sqrt{1-\tanh(\mu(y+y_0))}u_1^+ \\ \sqrt{1+\tanh(\mu(y+y_0))}u_2^+ \end{pmatrix}. && (3.65)
\end{aligned}$$

4 Discussion

Obtaining the low energy effective action (LEEA) is usually one of the objectives to study models with domain walls. There have been a number of works studying linear sigma models. Since nonlinear sigma model is often necessary to consider eight or more SUSY, we shall consider LEEA for nonlinear sigma model in this section. To illustrate an issue, we use component formalism here and leave the treatment in the superfield formalism for future work.

Let us take the component action of the $T^*\mathbf{C}P^1$ model in (3.58). In order to obtain a $\frac{1}{2}$ BPS wall solution, it has been found that one can consistently truncate [13] by taking $X^1, X^2 = 0$ to obtain a truncated model with φ and $X \equiv X^3$ whose range is $|X| \le \xi$ with $U = \xi/(\xi^2 - X^2)$

$$\int d^4x \mathcal{L}_{\rm truncated} = -\frac{1}{2}\int d^4x \left\{U\partial_\mu X\partial^\mu X + U^{-1}\partial_\mu\varphi\partial^\mu\varphi + \mu^2 U^{-1}\right\}. \tag{4.1}$$

If we assume the field configurations to depend on a single coordinate y, energy density in the truncated model can be rewritten as

$$\begin{aligned}\mathcal{E}_{\rm truncated} &= \frac{1}{2}\left[U\left(\partial_2 X\right)^2 + U^{-1}\left(\partial_2\varphi\right)^2 + \mu^2 U^{-1}\right] \\ &= \frac{1}{2}\left[U(\partial_2 X - \mu U^{-1})^2 + U^{-1}(\partial_2\varphi)^2\right] + \partial_2\left(\mu X\right), \end{aligned} \tag{4.2}$$

and one can obtain the $\frac{1}{2}$ BPS equation as

$$\frac{\partial\varphi}{\partial y} = 0, \qquad \frac{\partial X}{\partial y} = \mu U^{-1} = \mu\frac{\xi^2 - X^2}{\xi}. \tag{4.3}$$

The BPS wall solutions become

$$X_{\rm cl} = \xi\tanh\mu(y + y_0), \qquad \varphi_{\rm cl} = {\rm constant} = \varphi_0. \tag{4.4}$$

Without loss of generality, we can choose $y_0 = 0$ and $\varphi_0 = 0$.

To describe the field theory of fluctuations on the background, we decompose the fields in terms of mode functions which are usually defined by linearised equations of motion

$$X(x,y) = X_{\rm cl}(y) + \sum_n X_n(x)a_n(y), \qquad \varphi(x,y) = \varphi_{\rm cl}(y) + \sum_n \varphi_n(x)b_n(y), \tag{4.5}$$

where x denotes three-dimensional world volume coordinates of of domain wall. Among various bosonic modes, one can easily find massless modes corresponding to spontaneously broken global symmetry generators, namely the Nambu-Goldstone (NG) particles corresponding to translation and the $U(1)$ isometry

$$a_0(y) \equiv \frac{dX_{\rm cl}}{dy} = \frac{\mu\xi}{\cosh\mu y}, \qquad b_0(y) \equiv \left.\frac{d\varphi_{\rm cl}(y)}{d\varphi_0}\right|_{\varphi_0=0} = 1. \tag{4.6}$$

We are usually interested in LEEA of massless or nearly massless modes. If decoupling holds, massive modes give contributions suppressed by inverse powers of their masses after functional integration. In such a circumstance, we can obtain LEEA by retaining only the massless modes $X(x,y) \to X_{\rm cl}(y) + X_0(x)a_0(y)$, and $\varphi(x,y) = \varphi_{\rm cl}(y) + \varphi_0(x)b_0(y)$. After keeping only zero modes $X_0(x), \varphi_0(x)$ and integrating over y, we obtain a candidate of LEEA

$$\int d^3x \mathcal{L}_{\rm LE\, eff} = \int d^3x \left(\mathcal{L}_{{\rm kin}\varphi} + \mathcal{L}_{\rm kinX} + \mathcal{L}_{\rm pot} + \mathcal{E}_{\rm bkgr}\right), \tag{4.7}$$

$$\mathcal{L}_{\mathrm{kin}\varphi} = -\frac{\xi}{2}\left(\frac{2}{\mu} - \frac{4}{3}\mu X_0^2\right)(\partial_a \varphi_0)^2, \quad \mathcal{L}_{\mathrm{kinX}} = \frac{1}{2}\frac{\xi}{2\mu X_0^2}\log\left(1 - 4\mu^2 X_0^2\right)(\partial_a X_0)^2, \tag{4.8}$$

$$\mathcal{L}_{\mathrm{pot}} = -\frac{\xi\mu}{2}\left\{6 - \frac{4}{3}\mu^2 X_0^2 + \frac{2}{\mu X_0}\log\left(\frac{1-2\mu X_0}{1+2\mu X_0}\right) - \left(2 + \frac{1}{2\mu^2 X_0^2}\right)\log\left(1 - 4\mu^2 X_0^2\right)\right\}, \tag{4.9}$$

where index $a = 0, 1, 3$ denotes spacetime dimensions corresponding to the world volume of the wall, and $\mathcal{E}_{\mathrm{bkgr}} = 2\mu\xi$ is the energy density of the wall. Let us note that the $U(1)$ zero mode φ_0 is normalizable in spite of the constant wave function $b_0(y) \equiv 1$ in Eq.(4.6). This shows that the $U(1)$ zero mode is effectively localized because of the nonlinearity of the kinetic term, whereas the wave function $a_0(y)$ of the translation zero mode X_0 is manifestly localized. By examining the linearised equation of motion for the three-dimensional fields $X_0(x)$ and $\varphi_0(x)$, we find that the masses of these fields vanish. On the other hand, there are nonlinear interaction terms both of derivative and non-derivative types in Eqs. (4.7)-(4.9). There are two problems leading to the above result. One problem is that decoupling of massive modes requires to retain auxiliary fields of the remaining SUSY [24]. The other problem is the possible field redefinition which we now wish to examine.

In choosing a field variable for zero modes, we can take the celebrated Manton's method as a guiding principle. He has proposed to obtain (a part of) LEEA of zero modes [30], [16]. When a soliton has a moduli such as a translational collective coordinate, one promotes the collective coordinate to a field on the soliton world volume. Since the method presupposes a slow motion in the moduli space, the Manton's method should give LEEA of the zero mode at least up to two derivative terms correctly. The collective coordinates y_0 for the translation and φ_0 for the $U(1)$ isometry appear in the classical solution as in Eq. (4.4). Promoting the collective coordinates to the zero mode fields $y_0(x), \varphi_0(x)$ by the Manton's method, we obtain

$$X(x,y) = \xi \tanh \mu(y + y_0(x)) = X_{\mathrm{cl}}(y)|_{y_0=0} + \left.\frac{dX_{\mathrm{cl}}}{dy}(y)\right|_{y_0=0} y_0 + O(y_0^2) + \cdots, \tag{4.10}$$

$$\varphi(x,y) = \varphi_0(x) = \left.\frac{d\varphi_{\mathrm{cl}}}{dy}(y)\right|_{\varphi_0=0} \varphi_0(x). \tag{4.11}$$

As we have seen, the zero mode wave function is usually obtained by a derivative of the classical field configuration. This usual definition of zero mode field coincides with the Manton's method only when we choose the field variables as u, φ by a field redefinition $X \to u$

$$X = \xi \tanh u, \qquad v \equiv e^{\Omega} \equiv e^{u+i\varphi}. \tag{4.12}$$

We can identify u, φ as real and imaginary part of a chiral scalar field Ω, since a scalar multiplet of $\mathcal{N} = 2$ SUSY in three dimensions requires a complex scalar field. With u, φ, the truncated model is given by a massive $\mathbf{C}P^1$ nonlinear sigma model

$$\int d^4x \mathcal{L}_{\rm truncated} = -\frac{\xi}{2}\int d^4x \frac{1}{\cosh^2 u}\left((\partial_\mu u)^2 + (\partial_\mu \varphi)^2 + \mu^2\right)$$
$$= -\frac{\xi}{2}\int d^4x \frac{4}{(1+|v|^2)^2}\left(\partial_\mu v^* \partial^\mu v + \mu^2 v^* v\right) \tag{4.13}$$

where the Fubini-Study metric of $\mathbf{C}P^1$ can be recognised. This choice of field precisely corresponds to the truncated model with the $\mathcal{N}=1$ superfield Eqs. (2.19) and (2.20). BPS wall solution and the NG mode function for translation is given by

$$u_{\rm cl}(y) = \mu(y+y_0), \qquad a_{u0}(y) \equiv \frac{du_{\rm cl}(y)}{dy} = \mu. \tag{4.14}$$

After decomposing into modes $u(x,y) = u_{\rm cl}(y) + \sum u_n(x) a_{un}(y)$, retaining fields $u_0(x), \varphi_0(x)$ corresponding to zero modes and integrating over y, we find precisely a free massless complex scalar action

$$\int d^3x \mathcal{L}_{\rm LE\, eff} = -\frac{\mu\xi}{2}\int d^3x \left((\partial_a u_0)^2 + (\partial_a \varphi_0)^2\right) + \mathcal{E}_{\rm bkgr}. \tag{4.15}$$

We have computed all powers in the fields u_0, φ_0 without making approximations. Let us observe that the field redefinition from $X_0(x)$ to $u_0(x)$ is not just an ordinary field redefinition local in x. It involves all higher massive modes and functions of y in a complicated way as one can see by comparing mode expansions

$$X(x,y) = X_{\rm cl}(y) + \sum_n X_n(x) a_n(y) = \xi \tanh u(x,y) = \xi \tanh\left(u_{\rm cl}(y) + \sum_n u_n(x) a_{un}(y)\right). \tag{4.16}$$

We have no reason to believe that the LEEA in Eq. (4.15) can be obtained from Eq. (4.7) by a local field redefinition $u_0(x) = f\left(X_0(x)\right)$. Therefore it is important to choose the appropriate variable to define zero mode field. With the Manton's method, we are at least sure that the zero modes $u_0(x), \varphi_0(x)$ have no interactions as far as two derivative terms are concerned in conformity with the low energy theorems. LEEA of NG particles associated with walls can be obtained also by means of nonlinear realization [36], whose result is consistent with ours up to two derivatives as expected.

Let us also note that the choice of u, φ has also allowed the identification of these fields as a complex scalar field of the chiral scalar multiplet of $\mathcal{N}=2$ SUSY in three dimensions (4 supercharges). Thus the choice of the field variable is better for preserved SUSY which requires the two real scalar fields for zero mode to have the identical wave function.

When one wants to obtain the LEEA, Kähler Normal Coordinate (KNC) [38] may be useful to represent all modes. Using the KNC, any tensor can be expanded covariantly to keep the complex structure. Then, the general action described by Kähler potential and superpotential can be expanded, and on-shell action is obtained in terms of boson and fermion fields, which is described by the KNC geometrically. Applying the KNC with the real domain wall solution as the background and expanding the general action, it is easy to find the correspondence between mode equations for boson and fermion fields in all modes including massive modes. In Appendix D, we show such a correspondence.

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A Two $1/2$ SUSY conditions and Lorentz invariance

A.1 $1/2$ SUSY condition

In this Appendix, we derive the half SUSY condition (1.1), and find a $SU(2)_R$ transformation between the two models discribed in $\mathcal{N}=1$ superfield and the HSF, by examining the relation of $\frac{1}{2}$ SUSY conditions. We also show that models allowed by this condition are Lorentz invariant only for three dimensional spacetime.

In order to find SUSY conserved by BPS wall background, let us examine SUSY transformation of fermions in bosonic background. The SUSY transformation in $\mathcal{N}=2$ SUSY massive nonlinear sigma model in four dimensions ($D=4$) is obtained from the SUSY transformation in $\mathcal{N}=1$ SUSY nonlinear sigma models in six dimensions ($D=6$) with nontrivial dimensional reduction. The SUSY transformation of fermions in $D=6$ is given by

$$\delta\chi^a = \Gamma^M f_X^{ai}\partial_M\phi^X\varepsilon_i, \qquad (X=1,..,4n), \tag{A.1}$$

where $\Gamma^M(M=0,1,...,5)$ are the $D=6$ Dirac matrices, $f_X^{ai}(a=1,..,2n, i=1,2)$ are fielbeins of target hyper-Kähler manifold, transforming in the $(2n,2)$ representation of $Sp_n\times Sp_1$, and ε_i is a Sp_1-Majorana and Weyl spinor satisfying $\Gamma^{012345}\varepsilon^i=\varepsilon^i$. On-shell SUSY transformation of fermions in $D = 4$ $\mathcal{N} = 2$ SUSY nonlinear sigma models with potential terms can be obtained by the nontrivial dimensional reduction from $D = 6$ to $D = 4$ [13]

$$\frac{\partial \phi^X}{\partial x^4} = 0, \qquad \frac{\partial \phi^X}{\partial x^5} = \mu k^X, \tag{A.2}$$

where $k \equiv k^X \partial_X$ is a tri-holomorphic Killing vector, which is the same as Eq. (2.17), and μ is a real mass parameter.

Substituting the Eguchi-Hanson metric and corresponding Killing vector to the r.h.s. of Eqs. (A.1) and (A.2), and requiring the vanishing SUSY variation, we obtain the condition of SUSY configuration, after some algebra [10, 13],

$$\Gamma^\mu [\boldsymbol{\rho}_i{}^j \cdot \partial_\mu \mathbf{X} + \delta_i{}^j \cdot i U^{-1} \mathcal{D}_\mu \varphi] \varepsilon_j = -i\mu U^{-1} \Gamma^5 \varepsilon_i, \qquad (\mu = 0, ..., 3). \tag{A.3}$$

When we substitute the BPS equations (4.3) into the Eq.(A.3), we obtain the $\frac{1}{2}$ BPS condition for wall solution as

$$\Gamma^{25} (\rho^3)_i{}^j \varepsilon_j = -i\varepsilon_i. \tag{A.4}$$

If we choose the chiral representation of Dirac matrices such as

$$\Gamma^\mu = \gamma^\mu \otimes \tau^1, \qquad \Gamma^4 = \gamma^5 \otimes \tau^1, \qquad \Gamma^5 = i\mathbf{1}_4 \otimes \tau^2, \tag{A.5}$$

then spinor parameters can be reduced four component spinors like $(\varepsilon_i)^T = ((\varepsilon_i)^T, 0, 0, 0, 0)$, corresponding to two sets of Majorana spinors in $D = 4$. Moreover the Eq. (A.4) can be rewritten as

$$\sigma^2 \bar{\epsilon}^1 = i\epsilon_1, \qquad \sigma^2 \bar{\epsilon}^2 = -i\epsilon_2, \tag{A.6}$$

where ϵ_i are two Weyl spinors in $D = 4$, such as $(\varepsilon_i)^T \equiv ((\epsilon_i)^T, (\bar{\epsilon}^i)^T)$. This condition coincides with (2.22), if we identity $\epsilon = \epsilon_1$, for $\alpha = 0$ corresponding to the case of real parameters b and μ. From the Eq. (A.6) we can see that BPS wall solution conserves four SUSY out of eight SUSY in four dimensions.

A.2 $SU(2)_R$ transformation

We show that $\frac{1}{2}$ SUSY condtion (1.1) is related with (A.6) by an $SU(2)_R$ transformation. Let us take a $SU(2)$ transformation generated by

$$g_i{}^j \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \tag{A.7}$$

on the both sides of Eq.(A.4). Then we obtain the new condition

$$\Gamma^{25}(\rho^1)_i{}^j\varepsilon'_j = -i\varepsilon'_i. \tag{A.8}$$

given by new basis of spinor parameters $\varepsilon' \equiv g\varepsilon$. This can be rewritten, in two Weyl spinors in $D = 4$, as

$$\sigma^2\bar{\epsilon}'^1 = i\epsilon'_2 = i\epsilon'^1, \qquad \sigma^2\bar{\epsilon}'^2 = i\epsilon'_1 = -i\epsilon'^2, \tag{A.9}$$

where $(\varepsilon'_i)^T \equiv ((\epsilon'_i)^T, (\bar{\epsilon}'^i)^T)$. This new condition coincides with (1.1) and was used in subsection 3.3 to derive BPS equations. Therefore we can see that two $\frac{1}{2}$ SUSY conditions in this paper are related with each other by the $SU(2)_R$ transformation given by (A.7), which also relates between two models discussed in $\mathcal{N} = 1$ superfield and in HSF.

A possible identification of the FI term in $\mathcal{N} = 1$ language and $\mathcal{N} = 2$ (HSF) language is given as follows. In the $\mathcal{N} = 2$ language, the FI term is given as $\mathcal{L}_{FI} = \xi_{(ij)}D_v^{(ij)}$ after u integration and taking a normalisation . In this form, the $SU(2)_R$ covariance is manifest. Since the coefficients of the FI term $\xi^{(ij)}$ is $SU(2)_R$ triplet, it is represented as, for example, $\xi^{(ij)} = i(\xi^a\sigma^a\epsilon)^{(ij)}$ ($\xi_{(ij)} \equiv \overline{\xi^{(ij)}} = \epsilon_{ik}\epsilon_{jl}\xi^{(kl)}$) where ξ^a is a real parameter, while $D_v^{(ij)}$ is defined as $D_v^{(ij)} = i(D^a\sigma^a\epsilon)^{(ij)}$. It can be recognised as the FI term in the $\mathcal{N} = 1$ formalism by a relation $\xi^1 = \frac{b+\bar{b}}{2\sqrt{2}}$, $\xi^2 = \frac{i(b-\bar{b})}{2\sqrt{2}}$, $\xi^3 = -\frac{c}{2}$, $D^1 = \sqrt{2}\mathbf{Re}F$, $D^2 = \sqrt{2}\mathbf{Im}F$ and $D^3 = D$.

A.3 Lorentz symmetry

From (A.6), Majorana spinor parameters $\varepsilon_{\|i}$ ($\varepsilon_{\perp i}$) of SUSY conserved (broken) by the wall can be rewritten, by using projection operator, as

$$\begin{aligned} \varepsilon_{\|i} &\equiv \mathcal{P}\varepsilon_i \equiv \frac{1}{2}\left(1_4 - i\gamma^2\sigma^3{}_i{}^j\right)\varepsilon_j, \qquad \mathcal{P}^\dagger = \mathcal{P}, && \text{(A.10)} \\ \varepsilon_{\perp i} &\equiv (1_4 - \mathcal{P})\varepsilon_i \equiv \frac{1}{2}\left(1_4 + i\gamma^2\sigma^3{}_i{}^j\right)\varepsilon_j. && \text{(A.11)} \end{aligned}$$

Supercharges corresponding to conserved (broken) $\frac{1}{2}$ SUSY are given as, in Majorana representation,

$$\bar{\varepsilon}^i Q_i \equiv \bar{\varepsilon}^i_{\|}Q_{\|i} + \bar{\varepsilon}^i_{\perp}Q_{\perp i}, \quad Q_{\|i} \equiv (1_4 - \mathcal{P})Q_i, \quad Q_{\perp i} \equiv \mathcal{P}Q_i, \tag{A.12}$$

where the bar denotes the Dirac conjugate $\bar{\varepsilon} = \varepsilon^\dagger\gamma_0$. Original $\mathcal{N} = 2$ SUSY algebra in four dimensions is

$$\begin{aligned} \left[\bar{\varepsilon}^i Q_i, \bar{\eta}^j Q_j\right] &= (\varepsilon_i)^\dagger(\gamma_0)\left\{Q_i, (Q_j)^\dagger(\gamma_0)\right\}\eta_j \\ &= (\varepsilon_i)^\dagger(\gamma_0)\Big[2\gamma^\mu P_\mu\delta_i^j + \left\{1_4\Re(Z_{ij}) - \gamma^5\Im(Z_{ij})\right\}\Big]\eta_j. && \text{(A.13)} \end{aligned}$$

From Eqs. (A.10) and (A.12), the algebra of $\frac{1}{2}$ SUSY (4 SUSY) conserved by the wall becomes

$$\begin{aligned}\left[\bar{\varepsilon}^i Q_{\parallel i}, \bar{\eta}^j Q_{\parallel j}\right] &= (\varepsilon_{\parallel i})^\dagger(\gamma_0)\left\{Q_i, (Q_j)^\dagger(\gamma_0)\right\}\eta_{\parallel j} \\ &= (\varepsilon_i)^\dagger(\gamma_0)(1_4 - \mathcal{P})\left\{Q_i, (Q_j)^\dagger(\gamma_0)\right\}\mathcal{P}\eta_j \\ &= (\varepsilon_i)^\dagger(\gamma_0)\left[2\gamma^a P_a \delta_i^j + \{1_4 \Re(Z_{ij})\}\right]\mathcal{P}\eta_j, \quad (a = 0, 1, 3). \end{aligned} \tag{A.14}$$

We find in Eq. (A.14) that the momentum along y direction does not appear in the r.h.s. of the commutators of four supercharges conserved by the wall, but it appears in the r.h.s. of the commutators of supercharges broken and unbroken by the wall. The part (A.13) of algebra of SUSY charges in four dimensions conserved by the wall is equivalent to the $\mathcal{N} = 2$ SUSY algebra in three dimensions. Therefore the theory which maintains four supercharges conserved by the wall has Lorentz invariance of three dimensions, but not of four dimensions. We have thus understood this result, as due to the fact that $\frac{1}{2}$ SUSY condition of the wall breaks the four-dimensional Lorentz invariance preserving only three dimensional one.

B Hitchin coordinates

In this appendix, we give the $\mathcal{N} = 1$ superfield formulation of the massive HK sigma model on the asymptotically localy Euclidean (ALE) space, which inculdes $T^*\mathbf{C}P^1$. Then, we obtain the BPS wall solution in the case of $T^*\mathbf{C}P^1$. The multi-center metric of ALE may be used for the construction of an intersecting lumps solution [17] or a domain wall junction [8].

Let φ and $\mathbf{X} = (X^1, X^2, X^2)$ be coordinates of a four dimensional HK manifold with an $U(1)$ isometry of a constant shift of φ. The metric defined by [31]

$$ds^2 = U(\mathbf{X}) d\mathbf{X} \cdot d\mathbf{X} + U^{-1}(\mathbf{X})(d\varphi + \mathbf{A} \cdot d\mathbf{X})^2 \ , \tag{B.1}$$

$$\nabla \times \mathbf{A} = \nabla U \ , \tag{B.2}$$

is the multi-center ALE space if U is given by

$$U = \frac{1}{2} \sum_{i=1}^{k} \frac{1}{|\mathbf{X} - \mathbf{X}_i|} \ . \tag{B.3}$$

Here, k is the number of the centers, $\mathbf{X}_i$ is the position of the i-th center in the three dimensional space of $\mathbf{X}$, and $\mathbf{A} = (A_1, A_2, A_3)$ is a potential. The case of $k = 2$ corresponds to the Eguchi-Hanson space, $T^*\mathbf{C}P^1$.

We introduce complex coordinates using the method of Hitchin [32]. Here we follow Ref. [37]. For simplicity we take the parameter $\xi = 1$ in Eq.(3.60) here. First, we take a gauge of

$$
\begin{aligned}
A_1 &= 0 \ , \\
A_2 &= \frac{1}{2}\sum_i \frac{X^3 - X_i^3}{|\mathbf{X}-\mathbf{X}_i|(X^1 - X_i^1 - |\mathbf{X}-\mathbf{X}_i|)} \ , \\
A_3 &= \frac{1}{2}\sum_i \frac{-(X^2 - X_i^2)}{|\mathbf{X}-\mathbf{X}_i|(X^1 - X_i^1 - |\mathbf{X}-\mathbf{X}_i|)} \ .
\end{aligned}
\tag{B.4}
$$

Defining the complex coordinates $\varphi^i = (v, w)$ by

$$
v = X^2 + iX^3 \ , \tag{B.5}
$$

$$
w = Ce^{-i\varphi}\prod_{i=1}^{k}\sqrt{-b + b_i + \Delta_i} \ . \tag{B.6}
$$

the metric becomes

$$
ds^2 = U|dv|^2 + U^{-1}\left|\frac{dw}{w} - \delta dv\right|^2 \ . \tag{B.7}
$$

Here, we have defined

$$
\Delta_i \equiv |\mathbf{X}-\mathbf{X}_i| \ , \quad \delta \equiv \frac{1}{2}\sum_i \frac{(b - b_i) + \Delta_i}{\Delta_i (v - e_i)}, \tag{B.8}
$$

and

$$
b \equiv X^1 \ , \quad b_i \equiv X_i^1 \ , \quad e_i \equiv X_i^2 + iX_i^3 \ . \tag{B.9}
$$

In the derivation of (B.7), we have used

$$
\frac{dw}{w} = -Udb - id\varphi + \mathrm{Re}(\delta dv) \ . \tag{B.10}
$$

The components of the metric and its inverse in the coordinates $\varphi^i = (v, w)$ are obtained as

$$
g_{ij^*} = \begin{pmatrix} U + U^{-1}|\delta|^2 & -U^{-1}\frac{\delta}{w^*} \\ -U^{-1}\frac{\delta^*}{w} & U^{-1}\frac{1}{|w|^2} \end{pmatrix} \ , \tag{B.11}
$$

$$
g^{ij^*} = \begin{pmatrix} U^{-1} & U^{-1}\delta^* w^* \\ U^{-1}\delta w & |w|^2(U + U^{-1}|\delta|^2) \end{pmatrix} \ , \tag{B.12}
$$

respectively.

Since we know already that the scalar potential is given [see Eq. (3.58)] by

$$V = \mu^2 U^{-1} \ , \tag{B.13}$$

we find the superpotential

$$W = \mu v \ . \tag{B.14}$$

There are k isolated vacua given by $\mathbf{X} = \mathbf{X}_i$.

The BPS equation, $\partial_2 \varphi^i = -e^{i\alpha} g^{ij^*} \partial_{j^*} W^*$, for BPS walls in the general ALE space becomes

$$\partial_2 v = e^{i\alpha} \mu U^{-1} \ , \quad \partial_2 w = e^{i\alpha} \mu \delta w U^{-1} \ . \tag{B.15}$$

From now on, we concentrate on the Eguchi-Hanson space (two centers). Set the two centers as

$$(b_1, e_1) = (0, i), \qquad (b_2, e_2) = (0, -i) \ , \tag{B.16}$$

Let us set $X^1 = X^2 = 0$ and define $v = iX^3 \equiv iX$. Then,

$$U = \frac{1}{1 - X^2} \ , \quad \delta = iU, \tag{B.17}$$

and the BPS equations (B.15) become

$$i\partial_2 X = e^{i\alpha} \mu (1 - X^2) \ , \quad \partial_2 w = i e^{i\alpha} \mu w \ . \tag{B.18}$$

Now we must choose $\alpha = \frac{\pi}{2}$, and then we see that $|w| = 0$ and $\arg(w) =$ const. satisfy the second equation. The BPS domain wall solution is thus obtained as

$$-iv = X^3 = \tanh \mu (y + y_0) \ , \ \ -\arg(w) = \varphi = \varphi_0 \ . \tag{B.19}$$

C The HSF

C.1 Notation in the HSF

In this Appendix, we summarize our conventions in section 3, which are mostly the same as those of Refs. [25, 26] and [9]. Here we give only conventions related to the HSF.

Harmonic superspace is defined as $(x^\mu,\ \theta_i,\ \bar\theta^i, u_i^\pm)$ which is called the central basis. The $u_i^\pm$ is called the harmonic variable which parameterizes the coset $SU(2)_R/U(1)_r \sim S^2$. The superfield in the HSF is not defined in the central basis but in the subspace which is called the analytic subspace $\{\zeta_A, u_i^{\pm} | x_A^{\mu} = x^{\mu} - 2i\theta^{(i}\sigma^{\mu}\bar{\theta}^{j)} u_{(i}^{+}u_{j)}^{-},\ \theta^{+} = \theta^{i}u_i^{+},\ \bar{\theta}^{+} = \bar{\theta}^{i}u_i^{+},\ u_i^{\pm}\}$, where parentheses for indicies i, j mean symmetrization, for instance, $u_{(i}^{+}u_{j)}^{-} = (u_i^{+}u_j^{-} + u_j^{+}u_i^{-})/2$. Hypermultiplet and vector multiplet superfields are defined as the function in the analytic subspace as $q^{+}(\zeta_A, u)$ and $V^{++}(\zeta_A, u)$, respectively, which are called the analytic superfields.

To describe the real action in terms of the analytic superfield, the star conjugation must be introduced in addition to the usual complex conjugation. The complex conjugation rules for the cofficients in the harmonic expansions $f^{i_1 \cdots i_n}$ (see (C.14)), the Grassmann variable $\theta_{i\alpha}$ and the harmonic variable $u_i^{\pm}$ are defined as

$$\overline{f^{i_1\cdots i_n}} \equiv \bar{f}_{i_1\cdots i_n}, \qquad \overline{f_{i_1\cdots i_n}} = -\bar{f}^{i_1\cdots i_n}, \tag{C.1}$$

$$\overline{\theta_{i\alpha}} = \bar{\theta}^{i}_{\dot{\alpha}}, \qquad \overline{\theta^{i}_{\alpha}} = -\bar{\theta}_{\dot{\alpha} i}, \tag{C.2}$$

$$\overline{u^{+i}} = u_i^{-}, \qquad \overline{u_i^{+}} = -u^{-i}, \tag{C.3}$$

respectively. The star conjugation rules are defined as

$$(f^{i_1\cdots i_n})^{*} = f^{i_1\cdots i_n}, \tag{C.4}$$

$$(\theta^{i}_{\alpha})^{*} = \theta^{i}_{\alpha}, \tag{C.5}$$

$$(u^{+i})^{*} = u^{-i},\ \ (u_i^{+})^{*} = u_i^{-},\ \ (u^{-i})^{*} = -u^{+i},\ \ (u_i^{-})^{*} = -u_i^{+}, \tag{C.6}$$

$$(u_i^{\pm})^{**} = -u_i^{\pm}. \tag{C.7}$$

Note that the star conjugate acts only on the quantity having $U(1)_r$ charge. We write the combination of the complex and the star conjugation as

$$(\overline{q^{+}(\zeta_A, u)})^{*} \equiv \widetilde{q^{+}}(\zeta_A, u). \tag{C.8}$$

The combined conjugation rules are defined by

$$\widetilde{f^{i_1\cdots i_n}} = \overline{f^{i_1\cdots i_n}} \equiv \bar{f}_{i_1\cdots i_n}, \tag{C.9}$$

$$\widetilde{\theta^{+}} = \bar{\theta}^{+}, \quad \widetilde{\theta^{-}} = \bar{\theta}^{-}, \quad \widetilde{\bar{\theta}^{+}} = -\theta^{+}, \quad \widetilde{\bar{\theta}^{-}} = -\theta^{-}, \tag{C.10}$$

$$(\widetilde{u_i^{\pm}}) = u^{\pm i},\ \ (\widetilde{u^{\pm i}}) = -u_i^{\pm}. \tag{C.11}$$

The simple example of the real action is the free massless action of the Fayet-Sohnius hypermultiplet; $S = -\int d\zeta_A^{(-4)} du\ \widetilde{\phi^{+}} D^{++} \phi^{+}$ where D^{++} is defined by (3.4) with $Z = 0$. The action is real in the sense of ordinary complex conjugation $\bar{S} = S$. This property follows from the fact that $\widetilde{\widetilde{q^{+}}} = -q^{+}$.

C.2 The massive $T^*\mathbf{C}P^1$ model

In this Appendix, we show how to arrive at the on-shell action, and that it is described by on-shell bosonic fields $f^i_a(x_A)$, and their superpartners $\psi_a(x_A)$ and $\bar\varphi_a(x_A)$.

We first derive the equations of motion for components. The Grassmann expansions for analytic superfields are given by

$$\begin{aligned} q^+_a(\zeta_A,u) &= F^+_a+\sqrt{2}\theta^+\psi_a+\sqrt{2}\bar\theta^+\bar\varphi_a+i\theta^+\sigma^\mu\bar\theta^+A^-_{a\mu}+\theta^+\theta^+M^-_a+\bar\theta^+\bar\theta^+N^-_a \\ &\quad +\sqrt{2}\theta^+\theta^+\bar\theta^+\bar\chi^{--}_a+\sqrt{2}\bar\theta^+\bar\theta^+\theta^+\xi^{--}_a+\theta^+\theta^+\bar\theta^+\bar\theta^+D^{---}_a, \qquad (C.12)\\ V^{++}_{\mathrm{WZ}}(\zeta_A,u) &= \theta^+\theta^+\bar M_v+\bar\theta^+\bar\theta^+M_v-2i\theta^+\sigma^\mu\bar\theta^+V_\mu \\ &\quad +\sqrt{2}\theta^+\theta^+\bar\theta^+\bar\lambda^iu^-_i+\sqrt{2}\bar\theta^+\bar\theta^+\theta^+\lambda^iu^-_i+\theta^+\theta^+\bar\theta^+\bar\theta^+D^{(ij)}_vu^-_{(i}u^-_{j)}, \qquad (C.13)\end{aligned}$$

where $a=1,2$ is flavor index. Note that each component in the hypermultiplet analytic superfield (C.12) includes infinite series expanded by the harmonic variable (harmonic expansions), for instance,

$$F^+_a(\zeta_A,u)=\sum_{n=0}^{\infty}f^{(i_1\cdots i_{n+1}j_1\cdots j_n)}(x_A)u^+_{(i_1}\cdots u^+_{i_{n+1}}u^-_{j_1}\cdots u^-_{j_n)}. \qquad (C.14)$$

Thus, the hypermultiplet includes infinite many auxiliary fields in addition to physical fields. Similary, component in the vector multiplet can be expanded in general but infinite many auxiliary fields are eliminated by $U(1)$ gauge transformation (3.2). As a result, physical fields $M_v(x_A)$, $V_\mu(x_A)$, $\lambda^i(x_A)$ and auxiliary fields $D_v(x_A)^{(ij)}$ are left as in (C.13). Substituting (C.12) and (C.13) into (3.10)-(3.12), one obtains the equations of motion. The equation of motion (3.10) reads

$$\begin{aligned} \partial^{++}F^+_1=0,\quad \partial^{++}\psi_1=0,\quad \partial^{++}\bar\varphi_1 &= 0, \qquad (C.15)\\ \partial^{++}A^-_{1\mu}-2\partial^A_\mu F^+_1-2V_\mu F^+_2 &= 0, \qquad (C.16)\\ \partial^{++}M^-_1-\frac{\mu}{2}F^+_1+\bar M_vF^+_2 &= 0, \qquad (C.17)\\ \partial^{++}N^-_1+\frac{\mu}{2}F^+_1+M_vF^+_2 &= 0, \qquad (C.18)\\ \partial^{++}\bar\chi^{--}_1-\frac{\mu}{2}\bar\varphi_1+\bar M_v\bar\varphi_2-i\bar\sigma^\mu V_\mu\psi_2+\bar\lambda^-F^+_2-i\bar\sigma^\mu\partial^A_\mu\psi_1 &= 0, \qquad (C.19)\\ \partial^{++}\xi^{--}_1+\frac{\mu}{2}\psi_1+M_v\psi_2+i\sigma^\mu V_\mu\bar\varphi_2+\lambda^-F_2+i\sigma^\mu\partial^A_\mu\bar\varphi_1 &= 0, \qquad (C.20)\\ -\partial^\mu_AA^-_{1\mu}-\frac{\mu}{2}N^-_1+\frac{\mu}{2}M^-_1+\partial^{++}D^{---}_1+\bar M_vN^-_2+M_vM^-_2 & \\ -V^\mu A^-_{2\mu}-\bar\lambda^-\bar\varphi_2-\lambda^-\psi_2+D^{--}_vF^+_2 &= 0, \qquad (C.21)\end{aligned}$$

whereas equation (3.11) gives

$$\partial^{++}F^+_2=0,\quad \partial^{++}\psi_2=0,\quad \partial^{++}\bar\varphi_2 = 0, \qquad (C.22)$$

$$
\begin{aligned}
\partial^{++} A^{-}_{2\mu} - 2\partial^{A}_{\mu} F^{+}_{2} + 2V_{\mu} F^{+}_{1} &= 0, && \text{(C.23)}\\
\partial^{++} M^{-}_{2} - \frac{\mu}{2} F^{+}_{2} - \bar{M}_{v} F^{+}_{1} &= 0, && \text{(C.24)}\\
\partial^{++} N^{-}_{2} + \frac{\mu}{2} F^{+}_{2} - M_{v} F^{+}_{1} &= 0, && \text{(C.25)}\\
\partial^{++} \bar{\chi}^{--}_{2} - \frac{\mu}{2}\bar{\varphi}_{2} - \bar{M}_{v}\bar{\varphi}_{1} + i\bar{\sigma}^{\mu} V_{\mu}\psi_{1} - \bar{\lambda}^{-} F^{+}_{1} - i\bar{\sigma}^{\mu}\partial^{A}_{\mu}\psi_{2} &= 0, && \text{(C.26)}\\
\partial^{++} \xi^{--}_{2} + \frac{\mu}{2}\psi_{2} - M_{v}\psi_{1} - i\sigma^{\mu} V_{\mu}\bar{\varphi}_{1} - \lambda^{-} F^{+}_{1} + i\sigma^{\mu}\partial^{A}_{\mu}\bar{\varphi}_{2} &= 0, && \text{(C.27)}\\
-\partial^{\mu}_{A} A^{-}_{2\mu} - \frac{\mu}{2} N^{-}_{2} + \frac{\mu}{2} M^{-}_{2} + \partial^{++} D^{---}_{2} - \bar{M}_{v} N^{-}_{1} - M_{v} M^{-}_{1} & && \\
+V^{\mu} A^{-}_{1\mu} + \bar{\lambda}^{-}\bar{\varphi}_{1} + \lambda^{-}\psi_{1} - D^{--}_{v} F^{+}_{1} &= 0. && \text{(C.28)}
\end{aligned}
$$

As for constraint (3.12), we write down the relevant part only;

$$
\begin{aligned}
\widetilde{F^{+}_{1}} F^{+}_{2} - \widetilde{F^{+}_{2}} F^{+}_{1} + \xi^{++} &= 0, && \text{(C.29)}\\
\widetilde{F^{+}_{1}}\psi_{2} - F^{+}_{2}\varphi_{1} - \widetilde{F^{+}_{2}}\psi_{1} + F^{+}_{1}\varphi_{2} &= 0. && \text{(C.30)}
\end{aligned}
$$

Equations of motion (C.15)-(C.18) and (C.22)-(C.25) are purely kinematical. They eliminate the infinite set of auxiliary fields in the harmonic expansions. The solutions are given as (3.13)-(3.20). Substituting the component expansion (C.12) and (C.13) and on-shell condition (3.13)-(3.20) into the action (3.6), and integrating the Grassmann and the harmonic variable, we obtain

$$
\begin{aligned}
S \;=\; \int d^4 x_A \Bigg\{ & -\partial^{A}_{\mu} f^{i}_{a} \partial^{\mu}_{A} \bar{f}_{ai} - \frac{\mu^2}{4}(f^{i}_{a}\bar{f}_{ai})^2 + \frac{|\, (\mu/2) f^{i}_{a}\epsilon_{ab}\bar{f}_{bi} - \bar{\psi}_{a}\epsilon_{ab}\bar{\varphi}_{b}\,|^2}{f^{i}_{a}\bar{f}_{ai}} \\
& -i\bar{\psi}_{a}\bar{\sigma}^{\mu}\partial^{A}_{\mu}\psi_{a} - i\varphi_{a}\sigma^{\mu}\partial^{A}_{\mu}\bar{\varphi}_{a} - \frac{\mu}{2}(\psi_{a}\varphi_{a} + \bar{\psi}_{a}\bar{\varphi}_{a}) \\
& + \frac{(f^{i}_{a}\partial^{\mu}_{A}\epsilon_{ab}\bar{f}_{bi} - \partial^{\mu}_{A} f^{i}_{a}\epsilon_{ab}\bar{f}_{bi} - i\bar{\psi}_{a}\bar{\sigma}^{\mu}\epsilon_{ab}\psi_{b} - i\varphi_{a}\sigma^{\mu}\epsilon_{ab}\bar{\varphi}_{b})^2}{4 f^{i}_{a}\bar{f}_{ai}} \\
& -\frac{\lambda^{i}}{2}\epsilon_{ab}(\varphi_{a} f_{bi} - \psi_{a} f_{bi}) + \frac{\bar{\lambda}^{i}}{2}\epsilon_{ab}(\bar{\psi}_{a} f_{bi} - \bar{\varphi}_{a}\bar{f}_{bi}) - \frac{1}{3} D_{v(ij)}(\epsilon_{ab}\bar{f}^{(i}_{a} f^{j)}_{b} + \xi^{(ij)}) \Bigg\}, \qquad \text{(C.31)}
\end{aligned}
$$

where flavor indices are summed and $\epsilon_{12} = -1,\ \epsilon_{21} = 1$. The last line stands for constraints for bosons and fermions. It is found that full component action is described by the hypermultiplet components f^{i}_{a} and $\psi_{a},\ \bar{\varphi}_{a}$. Note that they are still subject to the constraint (3.26) and $\epsilon_{ab}(\varphi_{a} f_{bi} - \psi_{a} f_{bi}) = 0$. In the end of subsection 3.3, we briefly review how to solve the constraint (3.26), and rewrite the bosonic part (3.24) of the action (C.31) by independent variables as (3.58). It is found that the target metric in field space of the action (3.58) coincides with the Eguchi-Hanson metric [20, 28, 35].

C.3 BPS equations in the HSF

In this Appendix, it is shown that the BPS equations are given as (3.27)-(3.30). As mentioned in subsection 3.3, we are interested in the on-shell fermions $\psi_a(x_A)$ and $\bar{\varphi}_a(x_A)$. Thus, in the following, we first derive the SUSY transformations for these fermions. They can be read off from the SUSY transformations for the analytic superfields q_a^+. The SUSY transformations for the coordinates on the analytic basis are defined as

$$\begin{aligned} \delta_S x_A^\mu &= -2i(\epsilon^i\sigma^\mu\bar{\theta}^+ + \theta^+\sigma^\mu\bar{\epsilon}^i)u_i^-, \\ \delta_S\theta^+ &= \epsilon^i u_i^+, \quad \delta_S\bar{\theta}^+ = \bar{\epsilon}^i u_i^+, \quad \delta_S u_i^\pm = 0, \\ \delta_S x_5 &= 2i(\epsilon^i\theta^+ - \bar{\epsilon}^i\bar{\theta}^+)u_i^-, \\ \delta_S x_6 &= 2(\epsilon^i\theta^+ + \bar{\epsilon}^i\bar{\theta}^+)u_i^-, \end{aligned} \tag{C.32}$$

where the translations in x_5 and x_6 can be used to generate the central charge Z by the substitution $-i(\partial_5 + i\partial_6) \to \mu/2$. In general, μ is complex but we take it to be real as mentioned in section 3.1. To derive the SUSY transformations which preserve the Wess-Zumino gauge, we calculate the sum of the SUSY transformation and the compensating gauge transformation for the vector multiplet in the Wess-Zumino gauge,

$$\hat{\delta}V_{\rm WZ}^{++} = (\delta_S + \delta_G)V_{\rm WZ}^{++} = \delta_S V_{\rm WZ}^{++} + D^{++}\lambda, \tag{C.33}$$

where $\hat{\delta}$ denotes a sum of SUSY transformation δ_S and gauge transformation δ_G. The gauge parameter $\lambda(\zeta_A, u)$ is expanded in the component fields as

$$\begin{aligned} \lambda(\zeta_A, u) &= F_\lambda + \sqrt{2}\theta^+\psi_\lambda^- + \sqrt{2}\bar{\theta}^+\bar{\psi}_\lambda^- + \theta^+\theta^+\bar{M}_\lambda^{--} + \bar{\theta}^+\bar{\theta}^+M_\lambda^{--} \\ &\quad + i\theta^+\sigma^\mu\bar{\theta}^+A_{\lambda\mu}^{--} + \sqrt{2}\bar{\theta}^+\bar{\theta}^+\theta^+\xi_\lambda^{---} + \sqrt{2}\theta^+\theta^+\bar{\theta}^+\bar{\xi}_\lambda^{---} + \theta^+\theta^+\bar{\theta}^+\bar{\theta}^+D_\lambda^{(-4)}. \end{aligned} \tag{C.34}$$

Eq. (C.33) is defined as the new SUSY transformation. Substituting (C.34) into (C.33) and using (C.32) with the understanding $-i\partial_5 \to \mu/2$ lead to explicit form:

$$\begin{aligned} \hat{\delta}V_{\rm WZ}^{++} &= \hat{\delta}F_v^{++} + \sqrt{2}\theta^+\hat{\delta}\psi_v^+ + \sqrt{2}\bar{\theta}^+\hat{\delta}\bar{\psi}_v^+ + \theta^+\theta^+\hat{\delta}\bar{M}_v + \bar{\theta}^+\bar{\theta}^+\hat{\delta}M_v + i\theta^+\sigma^\mu\bar{\theta}^+\hat{\delta}V_\mu \\ &\quad + \sqrt{2}\bar{\theta}^+\bar{\theta}^+\theta^+\hat{\delta}\lambda^- + \sqrt{2}\theta^+\theta^+\bar{\theta}^+\hat{\delta}\bar{\lambda}^- + \theta^+\theta^+\bar{\theta}^+\bar{\theta}^+\hat{\delta}D_v^{--} \end{aligned} \tag{C.35}$$

where

$$\hat{\delta}F_v^{++}(x_A, u) = 0 + (\partial^{++}F_\lambda)(x_A, u), \tag{C.36}$$

$$\hat{\delta}\psi_v^+(x_A, u) = \sqrt{2}\epsilon^i u_i^+\bar{M}_v(x_A) - \sqrt{2}i\sigma^\mu\bar{\epsilon}^i u_i^+V_\mu(x_A) + (\partial^{++}\psi_\lambda^-)(x_A, u), \tag{C.37}$$

$$\hat{\delta}M_v(x_A,u) = \frac{1}{\sqrt{2}}\epsilon^i\lambda_i(x_A)+\sqrt{2}\epsilon^{(i}\lambda^{j)}(x_A)u^+_{(i}u^-_{j)}+(\partial^{++}M^{--}_\lambda)(x_A,u), \tag{C.38}$$

$$\hat{\delta}V_\mu(x_A,u) = -\frac{i}{2\sqrt{2}}(\bar{\epsilon}^i\bar{\sigma}_\mu\lambda_i-\epsilon^i\sigma_\mu\bar{\lambda}_i)(x_A)-\frac{i}{\sqrt{2}}(\bar{\epsilon}^{(i}\bar{\sigma}_\mu\lambda^{j)}-\epsilon^{(i}\sigma_\mu\bar{\lambda}^{j)})(x_A)u^+_{(i}u^-_{j)} -\frac{1}{2}(\partial^{++}A^{--}_{\lambda\mu}-2\partial^A_\mu F_\lambda)(x_A,u), \tag{C.39}$$

$$\hat{\delta}\lambda^-(x_A,u) = -\sqrt{2}i(\sigma^\mu\bar{\epsilon}^iu^-_i)\partial^A_\mu M_v(x_A)-\sqrt{2}\sigma^\nu\bar{\sigma}^\mu\epsilon^iu^-_i\partial^A_\mu V_\nu(x_A)+\sqrt{2}\epsilon^{(i}D^{jk)}_v(x_A)u^+_{(i}u^-_ju^-_{k)} -\frac{2\sqrt{2}}{3}\epsilon_jD^{(jk)}_v(x_A)u^-_k+(\partial^{++}\xi^{---}_\lambda+i\sigma^\mu\partial^A_\mu\bar{\psi}^-_\lambda)(x_A,u), \tag{C.40}$$

$$\hat{\delta}D^{--}_v(x_A,u) = \sqrt{2}i(\epsilon^{(i}\sigma^\mu\partial^A_\mu\bar{\lambda}^{j)}(x_A)-\bar{\epsilon}^{(i}\bar{\sigma}^\mu\partial^A_\mu\lambda^{j)}(x_A))u^-_{(i}u^-_{j)}+(\partial^{++}D^{(-4)}_\lambda-\partial^\mu_AA^{--}_{\lambda\mu})(x_A,u). \tag{C.41}$$

The last parenthesis for each equation means the contribution from the gauge transformation. The shifts from the Wess-Zumino gauge are represented by the fields with higher harmonic variables which do not appear in Eq. (C.13), and they are pulled back to the original one by taking the gauge parameter as

$$F_\lambda(x_A,u) = f_\lambda(x_A), \tag{C.42}$$

$$\psi^-_\lambda(x_A,u) = -\left(\sqrt{2}\epsilon^i\bar{M}_v(x_A)-\sqrt{2}i\sigma^\mu\bar{\epsilon}^iV_\mu(x_A)\right)u^-_i, \tag{C.43}$$

$$M^{--}_\lambda(x_A,u) = -\sqrt{2}\epsilon^{(i}\lambda^{j)}(x_A)u^-_{(i}u^-_{j)}, \tag{C.44}$$

$$A^{--}_{\lambda\mu}(x_A,u) = -\sqrt{2}i(\bar{\epsilon}^{(i}\bar{\sigma}_\mu\lambda^{j)}(x_A)-\epsilon^{(i}\sigma_\mu\bar{\lambda}^{j)}(x_A))u^-_{(i}u^-_{j)}, \tag{C.45}$$

$$\xi^{---}_\lambda(x_A,u) = -\sqrt{2}\epsilon^{(i}D^{jk)}_v(x_A)u^-_{(i}u^-_ju^-_{k)}, \tag{C.46}$$

$$D^{(-4)}_\lambda(x_A,u) = 0. \tag{C.47}$$

At this stage, we can obtain the SUSY transformations $\hat{\delta}$ in the Wess-Zumino gauge for on-shell fermions, which can be derived from the SUSY trandformation for hypermultiplets $\hat{\delta}q^+_a$. They are defined as

$$\hat{\delta}q^+_1 = (\delta_S+\delta_G)q^+_1=\delta_Sq^+_1-\lambda q^+_2, \tag{C.48}$$

$$\hat{\delta}q^+_2 = (\delta_S+\delta_G)q^+_2=\delta_Sq^+_2+\lambda q^+_1. \tag{C.49}$$

Substituting (C.12) and (C.42)-(C.47) into (C.48) and (C.49), the SUSY transformation $\hat{\delta}$ for on-shell fermions can be derived as

$$\hat{\delta}\psi_1(x_A,u) = \sqrt{2}\epsilon^iu^+_iM^-_1-\sqrt{2}i\sigma^\mu\bar{\epsilon}^iu^-_i(\partial^A_\mu F^+_1+V_\mu F^+_2) +\frac{i}{\sqrt{2}}\sigma^\mu\bar{\epsilon}^iu^+_iA^-_{1\mu}-\sqrt{2}\left(\frac{\mu}{2}F^+_1-\bar{M}_vF^+_2\right)\epsilon^iu^-_i, \tag{C.50}$$

$$\hat{\delta}\bar{\varphi}_1(x_A,u) = \sqrt{2}\bar{\epsilon}^iu^+_iN^-_1+\sqrt{2}i\bar{\sigma}^\mu\epsilon^iu^-_i(\partial^A_\mu F^+_1+V_\mu F^+_2)$$

$$
-\frac{i}{\sqrt{2}}\bar{\sigma}^\mu \epsilon^i u_i^+ A_{1\mu}^- + \sqrt{2}\left(\frac{\mu}{2}F_1^+ + M_v F_2^+\right)\bar{\epsilon}^i u_i^-, \tag{C.51}
$$

$$
\begin{aligned}
\hat{\delta}\psi_2(x_A,u) &= \sqrt{2}\epsilon^i u_i^+ M_2^- - \sqrt{2} i\sigma^\mu \bar{\epsilon}^i u_i^- (\partial_\mu^A F_2^+ - V_\mu F_1^+) \\
&\quad + \frac{i}{\sqrt{2}}\sigma^\mu \bar{\epsilon}^i u_i^+ A_{2\mu}^- - \sqrt{2}\left(\frac{\mu}{2}F_2^+ + \bar{M}_v F_1^+\right)\epsilon^i u_i^-,
\end{aligned} \tag{C.52}
$$

$$
\begin{aligned}
\hat{\delta}\bar{\varphi}_2(x_A,u) &= \sqrt{2}\bar{\epsilon}^i u_i^+ N_2^- + \sqrt{2} i\bar{\sigma}^\mu \epsilon^i u_i^- (\partial_\mu^A F_2^+ - V_\mu F_1^+) \\
&\quad - \frac{i}{\sqrt{2}}\bar{\sigma}^\mu \epsilon^i u_i^+ A_{2\mu}^- + \sqrt{2}\left(\frac{\mu}{2}F_2^+ - M_v F_1^+\right)\bar{\epsilon}^i u_i^-.
\end{aligned} \tag{C.53}
$$

Substituting on-shell condition (3.13)-(3.20) and using $u_i^+ u_j^- - u_i^- u_j^+ = \epsilon_{ij}$, we find

$$
\hat{\delta}\psi_1(x_A) = -\sqrt{2}\epsilon^i\left(\bar{M}_v f_{2i} - \frac{\mu}{2}f_{1i}\right) + \sqrt{2} i\sigma^\mu \bar{\epsilon}^i (\partial_\mu^A f_{1i} + V_\mu f_{2i}), \tag{C.54}
$$

$$
\hat{\delta}\bar{\varphi}_1(x_A) = -\sqrt{2}\bar{\epsilon}^i\left(M_v f_{2i} + \frac{\mu}{2}f_{1i}\right) - \sqrt{2} i\bar{\sigma}^\mu \epsilon^i (\partial_\mu^A f_{1i} + V_\mu f_{2i}), \tag{C.55}
$$

$$
\hat{\delta}\psi_2(x_A) = \sqrt{2}\epsilon^i\left(\bar{M}_v f_{1i} + \frac{\mu}{2}f_{2i}\right) + \sqrt{2} i\sigma^\mu \bar{\epsilon}^i (\partial_\mu^A f_{2i} - V_\mu f_{1i}), \tag{C.56}
$$

$$
\hat{\delta}\bar{\varphi}_2(x_A) = \sqrt{2}\bar{\epsilon}^i\left(M_v f_{1i} - \frac{\mu}{2}f_{2i}\right) - \sqrt{2} i\bar{\sigma}^\mu \epsilon^i (\partial_\mu^A f_{2i} - V_\mu f_{1i}). \tag{C.57}
$$

Finally, substituting half SUSY condition (1.1), we obtain the BPS equations (3.27)-(3.30).

D Mode expansions of boson and fermion

In this appendix, we show that there exists the correspondence of bosons and fermions in the mode expansions to all orders including massive modes. To this end, we extensively use the covariant expansion of the Lagrangian in the Kähler normal coordinates (KNC). First, we present a brief review of KNC [38], and then we discuss the mode expansion around the BPS domain wall background, generalising the result of Chibisov and Shifman [5].

D.1 Covariant expansion of nonlinear Lagrangian

The Lagrangian of $\mathcal{N}=1$ SUSY nonlinear sigma models is given by [19, 9]

$$
\mathcal{L} = \int d^4\theta\, K(\Phi,\Phi^\dagger) + \left[\int d^2\theta\, W(\Phi) + \text{c.c.}\right], \tag{D.1}
$$

where $\Phi^i(x,\theta,\bar{\theta})$ are chiral superfields which consist of complex scalar fields $A^i(x)$, Weyl fermions $\psi^i(x)$ and auxiliary complex scalar fields $F^i(x)$:

$$
\Phi^i(y,\theta) = A^i(y) + \sqrt{2}\theta\psi^i(y) + \theta\theta F^i(y)\,, \quad y^\mu = x^\mu + i\theta\sigma^\mu\bar{\theta}\,, \tag{D.2}
$$

and K is the Kähler potential and W is the superpotential. Field redefinitions of chiral superfields $\Phi^{i\prime} = f^i(\Phi)$ yield

$$A^{i\prime} = f^i(A)\,, \qquad \psi^{i\prime} = \frac{\partial f^i(A)}{\partial A^j}\psi^j, \tag{D.3}$$

and redefinitions of F^i. Therefore, the scalar fields A^i transform as coordinates in the target manifold, whereas the fermions ψ^i transform as a holomorphic tangent vector. Equations of motion for F^i are

$$F^i = \frac{1}{2}\Gamma^i{}_{jk}\psi^j\psi^k - g^{ij^*}\partial_{j^*}W^*, \tag{D.4}$$

where $\Gamma^i{}_{jk}(A, A^*) = g^{il^*}g_{jl^*,k^*}(A, A^*)$ is the connection. We denote differentiation by $g_{jl^*,k^*} \equiv \partial g_{jl^*}/\partial A^{k*}$.

After the elimination of the auxiliary fields, the Lagrangian of SUSY nonlinear sigma models is obtained as

$$\begin{aligned}\mathcal{L} = &-g_{ij^*}\partial_\mu A^i\partial^\mu A^{*j} - ig_{ij^*}\bar{\psi}^j\bar{\sigma}^\mu D_\mu\psi^i + \frac{1}{4}R_{ij^*kl^*}\psi^i\psi^k\bar{\psi}^j\bar{\psi}^l \\ &-g^{ij^*}D_iWD_{j^*}W^* - \frac{1}{2}D_iD_jW\psi^i\psi^j - \frac{1}{2}D_{i^*}D_{j^*}W^*\bar{\psi}^i\bar{\psi}^j\,,\end{aligned} \tag{D.5}$$

where D_i and D_μ are the covariant derivatives on the target space and their pull-back to the space-time, respectively:

$$\begin{aligned}&D_iW = \partial_iW(A)\,, \qquad D_iD_jW = \partial_i\partial_jW(A) - \Gamma^k{}_{ij}(A, A^*)\partial_kW(A)\,, \\ &D_\mu\psi^i = \partial_\mu\psi^i + \partial_\mu A^j\Gamma^i{}_{jk}(A, A^*)\psi^k\,.\end{aligned} \tag{D.6}$$

The Lagrangian is invariant under holomorphic field redefinitions, corresponding to holomorphic coordinate transformations in the target manifold.

We decompose scalar fields into the background fields $\varphi^i(x)$ and fluctuating fields $\pi^i(x)$ around them:

$$A^i(x) = \varphi^i(x) + \pi^i(x)\,. \tag{D.7}$$

There appear lots of non-covariant terms in the expansion of the Lagrangian in terms of π^i. To eliminate such non-covariant terms, we transform the fluctuation fields π^i to KNC fields by [38]

$$\omega^i = \sum_{n=1}^{\infty}\frac{1}{n!}[g^{ij^*}K_{,j^*i_1\cdots i_n}(z, z^*)]_\varphi\pi^{i_1}\cdots\pi^{i_n}\,, \tag{D.8}$$

where the index "φ" denotes a value evaluated at the background $A^i = \varphi^i$. Under holomorphic coordinate transformations of fluctuations $\pi^i \to \pi^{i\prime} = \pi^{i\prime}(\pi)$, the KNC ω^i transform as a holomorphic tangent vector, as well as the fermions:

$$\omega^i \to \omega^{i\prime} = \frac{\partial \pi^{i\prime}}{\partial \pi^j}\omega^j \,, \quad \psi^i \to \psi^{i\prime} = \frac{\partial \pi^{i\prime}}{\partial \pi^j}\psi^j \,. \tag{D.9}$$

In KNC, the Taylor expansion gives us a covariant expansion, because all non-covariant terms vanish. Then, using (D.9) we have covariant expansion of each term in the Lagrangian in general coordinates as follows:

$$\begin{aligned}
&g_{ij^*}(A,A^*) = g_{ij^*}|_\varphi + R_{ij^*kl^*}|_\varphi \omega^k \omega^{*l} + O(\omega^3)\,,\\
&g^{ij^*}(A,A^*) = g^{ij^*}|_\varphi + R^{ij^*}{}_{kl^*}|_\varphi \omega^k \omega^{*l} + O(\omega^3)\,,\\
&D_i W(A) = D_i W|_\varphi + D_j D_i W|_\varphi \omega^j + \frac{1}{2} D_{j_1} D_{j_2} D_i W|_\varphi \omega^{j_1}\omega^{j_2} + O(\omega^3)\,,\\
&\partial_\mu \pi^i(x) = D_\mu \omega^i(x) - \frac{1}{2}\partial_\mu \varphi^{*j}(x) R^i{}_{k_1 j^* k_2}|_\varphi \omega^{k_1}(x)\omega^{k_2}(x) + O(\omega^3)\,.
\end{aligned} \tag{D.10}$$

The covariant derivatives are defined by

$$\begin{aligned}
D_\mu \omega^i &= \partial_\mu \omega^i + \partial_\mu \varphi^j \Gamma^i{}_{jk}(\varphi,\varphi^*)\omega^k,\\
D_\mu \psi^i &= \partial_\mu \psi^i + \partial_\mu \varphi^j \Gamma^i{}_{jk}(\varphi,\varphi^*)\psi^k\,.
\end{aligned} \tag{D.11}$$

Using the expansion (D.10), the Lagrangian can be expanded in terms of KNC as

$$\mathcal{L} = \mathcal{L}^{(0)} + \mathcal{L}^{(1)} + \mathcal{L}^{(2)} + O(3)\,, \tag{D.12}$$

in which each order of the expansion can be calculated as

$$\begin{aligned}
\mathcal{L}^{(0)} &= -g_{ij^*}\partial_\mu \varphi^i \partial^\mu \varphi^{*j} - g^{ij^*} D_i W D_{j^*} W^*,\\
\mathcal{L}^{(1)} &= -g_{ij^*}(D_\mu \omega^i \partial^\mu \varphi^{*j} + \partial^\mu \varphi^i D_\mu \omega^{*j})\\
&\quad - g^{ij^*}[(D_k D_i W) D_{j^*} W^* \omega^k + D_i W (D_{l^*} D_{j^*} W)\omega^{*l}],\\
\mathcal{L}^{(2)} &= -g_{ij^*} D_\mu \omega^i D^\mu \omega^{*j}\\
&\quad - R_{ij^*kl^*}\left(\omega^k \omega^{*l}\partial_\mu \varphi^i \partial^\mu \varphi^{*j} - \frac{1}{2}\omega^i\omega^k \partial_\mu \varphi^{*j}\partial^\mu \varphi^{*l} - \frac{1}{2}\omega^{*j}\omega^{*l}\partial_\mu \varphi^i \partial^\mu \varphi^k\right)\\
&\quad + [g^{mj^*} g^{in^*} R_{mn^* k_1 l_1^*} D_i W D_{j^*} W^* - g^{ij^*}(D_{k_1} D_i W)(D_{l_1^*} D_{j^*} W^*)]\omega^{k_1}\omega^{*l_1}\\
&\quad - \frac{1}{2} g^{ij^*}[(D_{k_1} D_{k_2} D_i W) D_{j^*} W^* \omega^{k_1}\omega^{k_2} + D_i W (D_{l_1^*} D_{l_2^*} D_{j^*} W^*)\omega^{*l_1}\omega^{*l_2}]\\
&\quad - i g_{ij^*}\bar\psi^j \bar\sigma^\mu D_\mu \psi^i - \frac{1}{2} D_i D_j W \psi^i \psi^j - \frac{1}{2} D_{i^*} D_{j^*} W^* \bar\psi^i \bar\psi^j\,.
\end{aligned} \tag{D.13}$$

All tensors and the connection are evaluated at the background fields φ, but we use the same notation with the evaluation at general fields A unless there is confusion.

D.2 Expansion around a BPS Domain Wall Background

We consider (a parallel configuration of) BPS domain walls as a background φ^i. Without loss of generality, we can take the spatial direction perpendicular to the walls as y. The BPS equation for domain walls in the covariant form is

$$\varphi^{i\prime} = -g^{ij^*}(\varphi, \varphi^*) D_{j^*} W^*(\varphi^*), \tag{D.14}$$

where the prime denotes the differentiation with respect to the spatial coordinate y.

On the BPS domain wall background, the covariant derivative D_μ in spacetime reduces to

$$D_\mu = \delta^2_\mu (\varphi^{i\prime} D_i + \varphi^{*i\prime} D_{i^*}) = -\delta^2_\mu (g^{ij^*} D_{j^*} W^* D_i + g^{ij^*} D_i W D_{j^*})\,, \tag{D.15}$$

where we have used the BPS equation (D.14). Acting the operator D_2 on the BPS equation (D.14), we obtain

$$\varphi^{i\prime\prime} = g^{ij^*} g^{lk^*} D_l W (D_{k^*} D_{j^*} W^*)\,, \tag{D.16}$$

in which we have used $D_k D_{j^*} W^* = \partial_k \partial_{j^*} W^* = 0$ and the metric compatibility $D_k g_{ij^*} = 0$.

We now show that $\mathcal{L}^{(1)}$ around the wall background, given by

$$\begin{aligned} \mathcal{L}^{(1)} &= -g_{ij^*}(D_2 \omega^i \varphi^{*j\prime} + \varphi^{i\prime} D_2 \omega^{*j}) \\ &\quad - g^{ij^*}[(D_k D_i W) D_{j^*} W^* \omega^k + D_i W (D_{l^*} D_{j^*} W) \omega^{*l}]\,, \end{aligned} \tag{D.17}$$

is a total derivative. This can be rewritten as

$$\begin{aligned} \mathcal{L}^{(1)} &= -\partial_2[g_{ij^*}(\omega^i \varphi^{*j\prime} + \varphi^{i\prime} \omega^{*j})] + g_{ij^*}(\omega^i \varphi^{*j\prime\prime} + \varphi^{i\prime\prime} \omega^{*j}) \\ &\quad - g^{ij^*}[(D_k D_i W) D_{j^*} W^* \omega^k + D_i W (D_{l^*} D_{j^*} W) \omega^{*l}]\,. \end{aligned} \tag{D.18}$$

Substituting (D.16) into (D.18), we find

$$\mathcal{L}^{(1)} = -\partial_2[g_{ij^*}(\omega^i \varphi^{*j\prime} + \varphi^{i\prime} \omega^{*j})]\,. \tag{D.19}$$

We thus obtain the expansion of Lagrangian around the BPS domain wall background, given by

$$\begin{aligned} \mathcal{L} &= \mathcal{L}^{(0)} + \partial_2(\cdots) \\ &\quad - g_{ij^*} D_\mu \omega^i D^\mu \omega^{*j} - g^{ij^*}(D_{k_1} D_i W)(D_{l_1^*} D_{j^*} W^*) \omega^{k_1} \omega^{*l_1} \\ &\quad + \frac{1}{2} R_{ij^*kl^*} \left(g^{mj^*} g^{nl^*} D_m W D_n W \omega^i \omega^k + g^{im^*} g^{kn^*} D_{m^*} W^* D_{n^*} W^* \omega^{*j} \omega^{*l} \right) \\ &\quad - \frac{1}{2} g^{ij^*}[(D_{k_1} D_{k_2} D_i W) D_{j^*} W^* \omega^{k_1} \omega^{k_2} + D_i W (D_{l_1^*} D_{l_2^*} D_{j^*} W^*) \omega^{*l_1} \omega^{*l_2}] \\ &\quad - i g_{ij^*} \bar\psi^j \bar\sigma^\mu D_\mu \psi^i - \frac{1}{2} D_i D_j W \psi^i \psi^j - \frac{1}{2} D_{i^*} D_{j^*} W^* \bar\psi^i \bar\psi^j + O(3)\,, \end{aligned} \tag{D.20}$$

where we have used the BPS equation (D.14).

Next, we discuss the linearized equation of motion for fluctuation. We denote the coordinates along the wall by $x^a = (t, x, z)$. The linearized equations of motion can be derived from (D.20). The equations of motion for ω^{*i} read

$$
\begin{aligned}
&0 = D_a D^a \omega^i + D_2 D^2 \omega^i - g^{ij^*} g^{lm^*} (D_k D_l W)(D_{m^*} D_{j^*} W^*)\omega^k \\
&+[g^{ij^*} g^{pm^*} g^{kn^*} R_{pj^*kl^*} D_{m^*} W^* D_{n^*} W^* - g^{ij^*} g^{nm^*} D_n W (D_{m^*} D_{j^*} D_{l^*} W^*)]\omega^{*l},
\end{aligned}
\tag{D.21}
$$

where we have used the equation $g_{ij^*} D_\mu \omega^i D^\mu \omega^{*j} = -\omega^{*j} g_{ij^*} D_\mu D^\mu \omega^i + \partial_\mu(\omega^{*j} g_{ij^*} D^\mu \omega^i)$. The equations of motion for $\bar{\psi}^i$ and ψ^i read

$$
\begin{aligned}
0 &= -i\bar{\sigma}^a D_a \psi^i - i\bar{\sigma}^2 D_2 \psi^i - g^{ij^*}(D_{j^*} D_{k^*} W^*)\bar{\psi}^k \,, \\
0 &= -i\sigma^a D_a \bar{\psi}^i - i\bar{\sigma}^2 D_2 \bar{\psi}^i - g^{ji^*}(D_j D_k W)\psi^k,
\end{aligned}
\tag{D.22}
$$

respectively. In the case of flat target space, equations $R_{ij^*kl^*} = 0$, $D_\mu = \partial_\mu$ and $D_i = \partial_i$ hold, and therefore Eqs. (D.21) and (D.22) reduce to the linearized equation of motion [5] generalized to an arbitrary number of components.

Now let us decompose complex fields into real fields. First, set boson fields $\omega^i \equiv \omega^i_R + i\omega^i_I$. Then Eq. (D.21) is decomposed into

$$
\begin{aligned}
0 &= D_a D^a \omega^i_R - [-\delta^i_l (D_2)^2 + g^{ij^*} g^{km^*}(D_k D_l W)(D_{m^*} D_{j^*} W^*) \\
&\quad -g^{ij^*} g^{pm^*} g^{kn^*} R_{pj^*kl^*} D_{m^*} W^* D_{n^*} W^* + g^{ij^*} g^{nm^*} D_n W (D_{m^*} D_{j^*} D_{l^*} W^*)]\omega^l_R \\
&\equiv D_a D^a \omega^i_R - \mathcal{A}^i{}_l \omega^l_R \,, \\
0 &= D_a D^a \omega^i_I - [-\delta^i_l (D_2)^2 + g^{ij^*} g^{km^*}(D_k D_l W)(D_{m^*} D_{j^*} W^*) \\
&\quad +g^{ij^*} g^{pm^*} g^{kn^*} R_{pj^*kl^*} D_{m^*} W^* D_{n^*} W^* - g^{ij^*} g^{nm^*} D_n W (D_{m^*} D_{j^*} D_{l^*} W^*)]\omega^l_I \\
&\equiv D_a D^a \omega^i_I - \mathcal{B}^i{}_l \omega^l_I \,,
\end{aligned}
\tag{D.23}
$$

where we have defined two matrix-operators $\mathcal{A}^i{}_j$ and $\mathcal{B}^i{}_j$. Second, let us decompose the Weyl fermions into sets of real fermions as $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \equiv \psi_R + i\psi_I$. Then their complex conjugates are

$$
\sigma^2 \bar{\psi} = \begin{pmatrix} -i\bar{\psi}^{\dot{2}} \\ i\bar{\psi}^{\dot{1}} \end{pmatrix} = i \begin{pmatrix} \bar{\psi}_{\dot{1}} \\ \bar{\psi}_{\dot{2}} \end{pmatrix} = i \begin{pmatrix} \psi_1^* \\ \psi_2^* \end{pmatrix} = i(\psi_R - i\psi_I) \,.
\tag{D.24}
$$

After some calculation, it is found that Eq. (D.22) is decomposed into

$$
\begin{aligned}
0 &= -i\hat{D}\psi^i_R - [\delta^i_k D_2 + g^{ij^*}(D_{j^*} D_{k^*} W^*)]\psi^k_I \equiv -i\hat{D}\psi^i_R - (\mathcal{O}_R)^i{}_k \psi^k_I \,, \\
0 &= -i\hat{D}\psi^i_I - [-\delta^i_k D_2 + g^{ij^*}(D_{j^*} D_{k^*} W^*)]\psi^k_I \equiv -i\hat{D}\psi^i_I - (\mathcal{O}_I)^i{}_k \psi^k_R \,,
\end{aligned}
\tag{D.25}
$$

where we have defined the matrix-operators $\mathcal{O}_R$ and $\mathcal{O}_I$, and $\hat{D}$ is defined by

$$
\hat{D}_\alpha{}^\beta = \sigma^2{}_{\alpha\dot{\beta}} \bar{\sigma}^{a\dot{\beta}\beta} D_a = -\sigma^a{}_{\alpha\dot{\beta}} \bar{\sigma}^{2\dot{\beta}\beta} D_a \,.
\tag{D.26}
$$

D.3 Mode expansions

Finally, we show the correspondence of boson and fermion in the mode expansion. To this end, we show that the remarkable relations between the matrix-operators for bosons and fermions, defined in Eqs. (D.23) and (D.25), hold:

$$(\mathcal{O}_R)^i{}_k(\mathcal{O}_I)^k{}_j = \mathcal{B}^i{}_j\,, \quad (\mathcal{O}_I)^i{}_k(\mathcal{O}_R)^k{}_j = \mathcal{A}^i{}_j\,. \tag{D.27}$$

These can be shown as follows:

$$(\mathcal{O}_R)^i{}_k(\mathcal{O}_I)^k{}_j = -\delta^i_j(D_2)^2 + g^{im^*}g^{kl^*}(D_kD_mW)(D_{l^*}D_{j^*}W^*) + D_2[g^{il^*}(D_{l^*}D_{j^*}W^*)]\,. \tag{D.28}$$

Using Eq. (D.15), the last term can be rewritten as

$$\begin{aligned}&D_2[g^{il^*}(D_{l^*}D_{j^*}W^*)]\\ &= -g^{mn^*}g^{il^*}D_{n^*}W^*(D_mD_{l^*}D_{j^*}W^*) - g^{il^*}g^{nm^*}D_nW(D_{m^*}D_{l^*}D_{j^*}W^*)\,.\end{aligned} \tag{D.29}$$

The term in the parenthesis can be calculated, to give

$$D_mD_{l^*}D_{j^*}W^* = R_{ml^*j^*}{}^{p^*}D_{p^*}W^* + D_{l^*}D_mD_{j^*}W^* = R_{ml^*j^*}{}^{p^*}D_{p^*}W^*\,, \tag{D.30}$$

in which the equation $D_mD_{j^*}W^* = \partial_m\partial_{j^*}W^* = 0$ has been used. We thus have shown the first equation in (D.27). The second one can be shown in the same way.

Using the operator relation (D.27), we discuss the mode expansion. Prepare sets of the mode functions $a^j_n(y)$ and $b^j_n(y)$ $(n = 1, 2, \cdots)$ satisfying

$$(\mathcal{O}_R)^i{}_j a^j_n(y) = \kappa_n b^i_n(y)\,, \quad (\mathcal{O}_I)^i{}_j b^j_n(y) = \lambda_n a^i_n(y)\,. \tag{D.31}$$

Then these functions satisfy

$$(\mathcal{O}_I\mathcal{O}_R)^i{}_j a^j_n(y) = (m_n)^2 a^i_n(y)\,, \quad (\mathcal{O}_R\mathcal{O}_I)^i{}_j b^j_n(y) = (m_n)^2 b^i_n(y)\,, \tag{D.32}$$

where we have defined $m_n = \sqrt{\kappa_n\lambda_n}$. Therefore $a_n \equiv \{a^i_n\}$ $(b_n \equiv \{b^i_n\})$ is an eigenvector of the matrix operator $\mathcal{O}_I\mathcal{O}_R$ $(\mathcal{O}_R\mathcal{O}_I)$ with an eigenvalue $(m_n)^2$. Using a complete set of a_n and b_n, the fluctuating fields can be expanded as

$$\begin{aligned}\omega^i_R &= \sum_n \sqrt{\lambda_n}a^i_n(y)\omega^{(n)}_R(x^a)\,, \quad \omega^i_I = \sum_n \sqrt{\kappa_n}b^i_n(y)\omega^{(n)}_I(x^a)\,,\\ \psi^i_R &= \sum_n \sqrt{\lambda_n}a^i_n(y)\psi^{(n)}_R(x^a)\,, \quad \psi^i_I = \sum_n \sqrt{\kappa_n}b^i_n(y)\psi^{(n)}_I(x^a)\,.\end{aligned} \tag{D.33}$$

The linearized equations of motion (D.23) and (D.25) reduce to

$$\begin{aligned}D_aD^a\omega^{(n)}_R - (m_n)^2\omega^{(n)}_R = 0\,, &\qquad D_aD^a\omega^{(n)}_I - (m_n)^2\omega^{(n)}_I = 0\,,\\ -i\hat{D}\psi^{(n)}_R - m_n\psi^{(n)}_I = 0\,, &\qquad -i\hat{D}\psi^{(n)}_I - m_n\psi^{(n)}_R = 0\,,\end{aligned} \tag{D.34}$$

which are equations of motion in the three dimensional effective field theory on the wall.

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